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Exercise Set 1

Throughout this exercise set, k denotes a field.

1. Let $\mathcal{C}$ be a category which has all finite products and let G be an object of $\mathcal{C}$. Suppose that for each object T of $\mathcal{C}$ the set $h_G(T) = \mathrm{Hom}_{\mathcal{C}}(T, G)$ is endowed with the structure of a group, and for each morphism $f \colon T \to T'$ the induced function $f^* \colon h_G(T') \to h_G(T)$ is a group homomorphism. Show that G naturally has the structure of a group object of $\mathcal{C}$.

2. Let k^s be the separable closure of k and let $G_k = \mathrm{Gal}(k^s/k)$ be the absolute Galois group of k. By a finite G_k-set we mean a finite set equipped with a continuous action of G_k. (Note: continuity simply means the action factors through $\mathrm{Gal}(k'/k)$ for some finite Galois extension k'/k.) For a finite étale k-algebra A, let $\Phi(A)$ be the set of k-algebra homomorphisms $A \to k^s$, equipped with its natural action of G_k. For a finite G_k-set I, let $\Psi(I)$ be the algebra of G_k-invariants of $(k^s)^I$, where G_k acts on $(k^s)^I$ through its action on both k^s and I. (Here we write R^I for the product of the ring R with itself I times; it can also be thought of as the ring of functions $I \to R$.) Show that Φ and Ψ are quasi-inverse functors (between the categories of finite étale k-algebras and finite G_k-sets).

3. We say that a k-algebra A is *connected* if it has no idempotents other than 0 and 1; this is equivalent to $\mathrm{Spec}(A)$ being connected. We say that A is *geometrically conntected* if $A \otimes_k \overline{k}$ is connected. (a) Suppose A is a connected k-algebra that admits a homomorphism $A \to k$. Show that A is geometrically connected. (b) Suppose A and B are k-algebras, A is connected, and B is geometrically connected. Show that $A \otimes_k B$ is connected. (c) Let $G = \mathrm{Spec}(A)$ be a finite group scheme over k. Define G° to be the connected component of G containing the identity. Show that G° is a subgroup of G.

4. (a) Let R be an $\mathbf{F}_p$-algebra. Show that $\mathrm{Aut}((\alpha_p)_R) = \mathbf{G}_m(R)$. In other words, show that $\underline{\mathrm{Aut}}(\alpha_p) = \mathbf{G}_m$, as functors. (b) Let G be the semi-direct product $\alpha_p \rtimes \mathbf{G}_m$, considered as a (non-commutative) group scheme over $\mathbf{F}_p$. Show that G_{red} is a non-normal subgroup of G. (You can take this to mean: $G_{\mathrm{red}}(R)$ is a subgroup of $G(R)$ for all $\mathbf{F}_p$-algebras R, and not a normal subgroup for some $\mathbf{F}_p$-algebra R.)

5. For an $\mathbf{F}_p$-algebra R, let $W_2(R) = R^2$, the set of ordered pairs in R. Define a group structure on $W_2(R)$ by

$$(x, y) + (x', y') = (x + x', y + y' + f(x, x'))$$

where f is the polynomial

$$f(X, Y) = \frac{(X+Y)^p - X^p - Y^p}{p}.$$

(a) Verify that $A \mapsto W_2(A)$ is a functor from $\mathbf{F}_p$-algebras to commutative groups. (b) Describe the group $W_2(\mathbf{F}_p)$. (c) Show that W_2 is an affine group scheme, i.e., construct a Hopf algebra A over $\mathbf{F}_p$ and a nautral isomorphism $W_2(R) = \mathrm{Hom}(A, R)$. (d) Let $W_2[F]$ be the kernel of the Frobenius map on W_2. Show that $W_2[F]$ is an extension of α_p by α_p. (e) Compute the Cartier dual of $W_2[F]$.

These are exercises for Math 679, taught in the Fall 2013 semester at the University of Michigan by Andrew Snowden.

6. Suppose k has characteristic p. For two finite group schemes G and H over k, let $\mathrm{Ext}^1(G, H)$ be the group of isomorphism classes of extensions of G by H. Compute $\mathrm{Ext}^1(G, H)$ for $G, H \in \{\mathbf{Z}/p\mathbf{Z}, \mu_p, \alpha_p\}$. (Start with the case where k is perfect and use Dieudonné modules, then try the general case.)