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Exercise Set 1

Throughout this exercise set, $k$ denotes a field.

1. Let $\mathcal{C}$ be a category which has all finite products and let $G$ be an object of $\mathcal{C}$. Suppose that for each object $T$ of $\mathcal{C}$ the set $h_G(T) = \text{Hom}_\mathcal{C}(T, G)$ is endowed with the structure of a group, and for each morphism $f: T \to T'$ the induced function $f^*: h_G(T') \to h_G(T)$ is a group homomorphism. Show that $G$ naturally has the structure of a group object of $\mathcal{C}$.

2. Let $k^s$ be the separable closure of $k$ and let $G_k = \text{Gal}(k^s/k)$ be the absolute Galois group of $k$. By a finite $G_k$-set we mean a finite set equipped with a continuous action of $G_k$. (Note: continuity simply means the action factors through $\text{Gal}(k'/k)$ for some finite Galois extension $k'/k$.) For a finite étale $k$-algebra $A$, let $\Phi(A)$ be the set of $k$-algebra homomorphisms $A \to k^s$, equipped with its natural action of $G_k$. For a finite $G_k$-set $I$, let $\Psi(I)$ be the algebra of $G_k$-invariants of $(k^s)^I$, where $G_k$ acts on $(k^s)^I$ through its action on both $k^s$ and $I$. (Here we write $R^I$ for the product of the ring $R$ with itself $I$ times; it can also be thought of as the ring of functions $I \to R$.) Show that $\Phi$ and $\Psi$ are quasi-inverse functors (between the categories of finite étale $k$-algebras and finite $G_k$-sets).

3. We say that a $k$-algebra $A$ is connected if it has no idempotents other than 0 and 1; this is equivalent to $\text{Spec}(A)$ being connected. We say that $A$ is geometrically connected if $A \otimes_k \overline{k}$ is connected. (a) Suppose $A$ is a connected $k$-algebra that admits a homomorphism $A \to k$. Show that $A$ is geometrically connected. (b) Suppose $A$ and $B$ are $k$-algebras, $A$ is connected, and $B$ is geometrically connected. Show that $A \otimes_k B$ is connected. (c) Let $G = \text{Spec}(A)$ be a finite group scheme over $k$. Define $G^0$ to be the connected component of $G$ containing the identity. Show that $G^0$ is a subgroup of $G$.

4. (a) Let $R$ be an $\mathbb{F}_p$-algebra. Show that $\text{Aut}((\alpha_p)_R) = G_m(R)$. In other words, show that $\text{Aut}(\alpha_p) = G_m$, as functors. (b) Let $G$ be the semi-direct product $\alpha_p \rtimes G_m$, considered as a (non-commutative) group scheme over $\mathbb{F}_p$. Show that $G_{\text{red}}$ is a non-normal subgroup of $G$. (You can take this to mean: $G_{\text{red}}(R)$ is a subgroup of $G(R)$ for all $\mathbb{F}_p$-algebras $R$, and not a normal subgroup for some $\mathbb{F}_p$-algebra $R$.)

5. For an $\mathbb{F}_p$-algebra $R$, let $W_2(R) = R^2$, the set of ordered pairs in $R$. Define a group structure on $W_2(R)$ by

$$(x, y) + (x', y') = (x + x', y + y' + f(x, x'))$$

where $f$ is the polynomial

$$f(X, Y) = \frac{(X + Y)^p - X^p - Y^p}{p}.$$  

(a) Verify that $A \mapsto W_2(A)$ is a functor from $\mathbb{F}_p$-algebras to commutative groups. (b) Describe the group $W_2(\mathbb{F}_p)$. (c) Show that $W_2$ is an affine group scheme, i.e., construct a Hopf algebra $A$ over $\mathbb{F}_p$ and a neutral isomorphism $W_2(R) = \text{Hom}(A, R)$. (d) Let $W_2[F]$ be the kernel of the Frobenius map on $W_2$. Show that $W_2[F]$ is an extension of $\alpha_p$ by $\alpha_p$. (e) Compute the Cartier dual of $W_2[F]$.

These are exercises for Math 679, taught in the Fall 2013 semester at the University of Michigan by Andrew Snowden.
6. Suppose $k$ has characteristic $p$. For two finite group schemes $G$ and $H$ over $k$, let $\text{Ext}^1(G, H)$ be the group of isomorphism classes of extensions of $G$ by $H$. Compute $\text{Ext}^1(G, H)$ for $G, H \in \{\mathbb{Z}/p\mathbb{Z}, \mu_p, \alpha_p\}$. (Start with the case where $k$ is perfect and use Dieudonné modules, then try the general case.)