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Lecture 7: Group schemes 3

The final lecture on group schemes is devoted entirely to Raynauds theorem: in mixed characteristic and low ramification, a group scheme is determined by its generic fiber. The proof proceeds by first reducing the statement to a special class of group schemes, the Raynaud F-module schemes. Next, these schemes are classified; this is the meat of the argument. (In lecture, I didn’t have time to prove the important congruence relations, but they are covered in the notes here.) Finally, one verifies the theorem for Raynaud F-module schemes using the classification result.

A good reference for today is Tates article “Finite flat group schemes” in the book “Modular forms and Fermats last theorem” (MR1638478), especially section 4.

1 Finite flat group schemes

Up until now, we have considered finite commutative group schemes over a field. We now work over a more general base scheme $S$, which we assume to be affine and noetherian; write $S = \text{Spec}(R)$. To have an analogous theory, we only consider flat group schemes. A finite flat $R$-module is projective, and so the coordinate rings of our group schemes will be projective $R$-modules.

Much of what we have previously done carries over:

- Finite flat group schemes over $S$ correspond to Hopf algebras over $R$ which are finitely generated and projective.
- We define the order of a finite flat group scheme as the rank of the corresponding Hopf algebra. In general, this is a locally constant function on $S$, but if $S$ is connected it can be treated as a single number.
- Quotients work: if $G$ and $H$ are finite and flat and $H$ is a closed subgroup of $G$ then $G/H$ exists and is finite and flat of the expected order.
- The classification of tame group schemes is similar: assuming $S$ is connected, the category of finite flat commutative tame group schemes is equivalent to the category of finite $\pi_1(S, s)$-modules, where $s$ is a geometric point of $S$. If $R$ is the ring of integers in a finite extension $K$ of $\mathbb{Q}_p$, then $\pi_1(S, s)$ is the Galois group of the maximal unramified extension of $K$.
- Assume $R$ is henselian local (e.g., complete local). Then one can treat the connected–tame sequence in the same manner. If $G = \text{Spec}(A)$ is a group scheme, then $A$ is semi-local and thus decomposes as $A = \prod A_i$. Once again, $G^0 = \text{Spec}(A_0)$, where the counit factors through $A_0$. One then has $G^\text{et} = G/G^0$.
- If the order of $G$ is invertible on $S$ then $G$ is tame.
- Cartier duality works in the same way.

These are notes for Math 679, taught in the Fall 2013 semester at the University of Michigan by Andrew Snowden.
2 Raynaud’s theorem

2.1 Statement of theorem

Let $K/\mathbb{Q}_p$ be a finite extension, let $R$ be its ring of integers in $K$, let $k$ be the residue field of $R$, and let $e$ be the ramification index of $K/\mathbb{Q}_p$. A prolongation of a finite group scheme $G_0/K$ is a finite flat group scheme $G/R$ equipped with an isomorphism $G_K \to G_0$.

**Theorem 1** (Raynaud). Suppose $e < p - 1$. Let $G_0$ be a finite commutative group scheme over $K$. Then any two prolongations of $G_0$ to $R$ are isomorphic.

**Remark 2.** This theorem is clearly not true without the hypothesis on $e$: indeed, if $K$ has the $p$th roots of unity then $\mu_p$ and $\mathbb{Z}/p\mathbb{Z}$ are isomorphic over $K$, and thus both prolongations of $\mathbb{Z}/p\mathbb{Z}$, but are not isomorphic over $R$ (as one is tale and one is connected).

We say that a group scheme $G_0$ over $K$ has property UP (unique prolongation) if any two prolongations of $G_0$ to $R$ are isomorphic. Raynaud’s theorem says UP holds for all $G_0$ if $e < p - 1$.

The proof of Raynaud’s theorem can be divided into three steps. First we show that if $G_0$ is an extension and UP holds for the sub and quotient then it holds for $G_0$. This means it suffices to prove UP for simple groups. We then classify the simple groups and their prolongations. This is the most involved step. Finally, we check by hand that UP holds when $e < p - 1$.

2.2 Prolongations

Let $G_0 = \text{Spec}(A_0)$ be a finite commutative group scheme over $K$. Prolongations of $G_0$ correspond to finite $R$-subalgebras $A$ of $A_0$ which are closed under comultiplication and span $A_0$ over $K$ (these conditions imply that $A$ is closed under the antipode). We partially order prolongations using inclusion on rings.

**Proposition 3.** Two prolongations have an inf and a sup.

**Proof.** If $A$ and $A'$ are the rings of two prolongations then $AA'$ is clearly closed under comultiplication, and thus a prolongation greater than each; it is clearly the unique minimal one, and thus the inf. Sups follow from Cartier duality. □

**Proposition 4.** If $G_0$ has a prolongation then it has a maximal one $G^+$ and a minimal one $G^-$. 

**Proof.** Since $A_0$ is a finite tale $K$-algebra, it has a maximal order. It follows that any ascending chain of prolongations stabilizes, and so there exists a maximal prolongation. The minimal one follows from Cartier duality. □

We thus see that UP holds for $G_0$ if and only if the natural map $G^+ \to G^-$ is an isomorphism (or, in terms of rings, $A^+ = A^-$). In particular, one can check UP by passing to an extension of $K$.

Suppose now that

$$0 \to G'_0 \to G_0 \to G''_0 \to 0$$

is a short exact sequence of group schemes and that $G_0$ admits a prolongation $G$. Then the scheme-theoretic closure $G'$ of $G'_0$ in $G$ is a prolongation of $G'_0$, and the quotient $G'' = G/G'$ is a prolongation of $G''_0$. Furthermore, if $H$ is a second prolongation of $G_0$ which is less than $G$ (i.e.,
there is a map $G \to H$), then clearly $H'$ is less than $G'$ and $H''$ is less than $G''$, and the following diagram commutes

$$
\begin{array}{c}
0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow H' \rightarrow H \rightarrow H'' \rightarrow 0
\end{array}
$$

Thus, if the maps $G' \to H'$ and $G'' \to H''$ are isomorphisms, so is $G \to H$. This proves the following:

**Proposition 5.** In the above situation, if $G'_0$ and $G''_0$ satisfy UP then so does $G_0$.

### 2.3 Raynaud $F$-module schemes

Let $G_0/K$ be a simple group scheme of $p$-power order, and let $V = G_0(K)$, and irreducible $F_p$-representation of the Galois group. Then $F = \text{End}_{G_0}(V)$ is a finite extension of $F_p$, and we can regard $V$ as an absolutely irreducible $F$-linear representation of $G_K$.

Suppose now that $k$ is algebraically closed, e.g., pass to $K^\text{un}$. Then $G_K$ is an extension of a tame part $I^t$, which is abelian, by a wild part $I^w$, which is pro-$p$. Since $I^w$ is pro-$p$, it must fix a non-zero vector in $V$; thus $V^{I^w}$ is non-zero. But $I^w$ is normal in $G_K$, and so $V^{I^w}$ is a subrepresentation of $V$. Since $V$ is simple, this implies $V = V^{I^w}$, i.e., $I^w$ acts trivially. Thus the action of $G_K$ on $V$ factors through $I^t$. Since $V$ (regarded as an $F$-vector space) is an absolutely irreducible representation of an abelian group, it necessarily has dimension 1, i.e., $\dim_F(V) = 1$.

An $F$-module scheme (over $R$ or $K$) is a group scheme $G$ equipped with a ring homomorphism $F \to \text{End}(G)$. If $G = \text{Spec}(A)$, we write $[t]$ for the map $A \to A$ induced by $t \in F$. A Raynaud $F$-module scheme $G$ is an $F$-module scheme of the same order as $F$; thus it is an $F$-module scheme such that $G(K)$ is one-dimensional over $F$. The above discussion proves the following:

**Proposition 6.** Suppose $k$ is algebraically closed and $G_0/K$ is simple of $p$-power order. Then $G_0$ is canonically a Raynaud $F$-module scheme (for some $F$).

**Proposition 7.** (No hypothesis on $k$.) Suppose UP holds for every Raynaud $F$-module scheme over $K^\text{un}$. Then UP holds for all finite order group schemes over $K$.

**Proof.** Let $G_0/K$ be given. We can check UP for $G_0$ over $K^\text{un}$, so we can assume $k$ is closed. We have shown that UP can be deduced if it is known for the outer groups in an extension. It thus suffices to treat the case where $G_0$ is simple. If $G_0$ is $p$-power order then it is a Raynaud $F$-module scheme, and satisfies UP by hypothesis. If $G_0$ is prime-to-$p$ then UP is automatic, as any prolongation is tame. \hfill \Box

Suppose now that $G_0$ is a Raynaud $F$-module scheme over $K$. The action of $F$ need not extend to an arbitrary prolongation of $G_0$. However, it necessarily extends to the maximal and minimal prolongations since they are unique. (For $t \in F^x$, the map $[t]$ of $A_0$ is an automorphism of Hopf algebras, and therefore must carry $A^+$ into itself.) Thus $G^+$ and $G^-$ are Raynaud $F$-module schemes, and the map $G^+ \to G^-$ respects the $F$-structure.

### 3 Analysis of Raynaud $F$-module schemes

#### 3.1 Set-up

We now analyze Raynaud $F$-module schemes. We fix the finite field $F$, and write $q = p^r$ for its order. We assume that $k$ contains the $q - 1$ roots of unity (i.e., $F$ embeds into $k$).
A character \( \mathbf{F}^\times \to R^\times \) is fundamental if the composite map \( \mathbf{F}^\times \to k^\times \) extends to an embedding of fields. If \( \chi \) is a fundamental character, then any other one is of the form \( \chi^p \) for some \( k \). Enumerate the fundamental characters as \( (\chi_i)_{i \in \mathcal{I}} \), for some index set \( \mathcal{I} \). For \( i \in I \), define \( i + 1 \) by \( \chi_{i+1} = \chi_i^p \). Then \( \mathcal{I} \) is a torsor for \( \mathbb{Z}/p\mathbb{Z} \).

An arbitrary character \( \mu: \mathbf{F}^\times \to R^\times \) can be expressed as a product \( \prod_{i \in I} \chi_i^{a_i} \) with \( a_i \in \mathbb{Z} \). This product is unique if we impose the conditions that \( 0 \leq a_i \leq p-1 \) and not all the \( a_i \) are equal to 0. We write \( \mu(i) \) in place of \( a_i \). Note that if \( \mu \) is the trivial character then \( \mu(i) = p-1 \) for all \( i \).

### 3.2 Initial analysis

Let \( G = \text{Spec}(A) \) be a Raynaud \( \mathbf{F} \)-module scheme over \( R \). Then \( G_{\overline{K}} \) is isomorphic to the constant group scheme \( \mathbf{F} \) over \( \overline{K} \). We fix an isomorphism of \( A_{\overline{K}} \) with the algebra of functions \( \mathbf{F} \to \overline{K} \). For a character \( \mu: \mathbf{F}^\times \to K^\times \), we let \( \epsilon_{\mu} \) be the function \( \mathbf{F} \to K \) extending \( \mu \) with and with \( \epsilon_{\mu}(0) = 0 \). Thus \( \epsilon_{\mu} \) is an element of \( A_{\overline{K}} \). We put \( \epsilon_i = \epsilon_{\chi_i} \).

Let \( I \) be the augmentation ideal of \( A \). Since \( \mathbf{F}^\times \) is a finite group whose order is invertible in \( R \) and all characters of \( \mathbf{F}^\times \) are defined over \( R \) (since \( k \) contains the \( q-1 \) roots of unity), we can decompose \( I \) as a sum \( \bigoplus \mu I_{\mu} \), where the sum is over the characters \( \mu \) of \( \mathbf{F}^\times \), and \( I_{\mu} \) is the \( R \)-submodule of \( I \) consisting of elements \( x \) such that \( [t]x = \mu(t)x \) for all \( t \in \mathbf{F}^\times \). Clearly, \( I_{\mu} \otimes_R \overline{K} \) is spanned by \( \epsilon_{\mu} \), and so \( I_{\mu} \) is a free \( R \)-module of rank 1.

For each \( i \in \mathcal{I} \), choose a non-zero element \( X_i \) of \( I_{\chi_i} \). Then \( X_i = c_i \epsilon_i \) for some \( c_i \in \overline{K}^\times \). Since \( X_i^p \) clearly belongs to \( I_{\chi_{i+1}} \), we have \( X_i^p = \delta_i X_{i+1} \) for some \( \delta_i \in R \). As \( \epsilon_i^p = \epsilon_{i+1} \), we find \( \delta_i = c_i^p / c_{i+1} \).

For a character \( \mu = \prod \chi_i^{\mu(i)} \) of \( F \), let \( X_{\mu} \) be the product \( \prod X_i^{\mu(i)} \), an element of \( I_{\mu} \). Note that what we might call \( e_{\mu} \), namely \( \prod \epsilon_i^{\mu(i)} \), is simply \( \epsilon_{\mu} \).

Let \( G^\vee = \text{Spec}(B) \) be the Cartier dual of \( G \), so that \( B \) is the \( R \)-linear dual of \( A \). Then \( B_{\overline{K}} \) is naturally identified with the group algebra \( \overline{K}[\mathbf{F}] \). We write \( \{t\} \) for the element of \( \overline{K}[\mathbf{F}] \) corresponding to \( t \in \mathbf{F} \). Note that \( \{t\} \{s\} = \{t+s\} \) (multiplication in the group algebra), but \( \{t\} \{s\} = \{ts\} \) (the \( \mathbf{F} \)-module structure). Note also that the pairing \( \langle , \rangle : A_{\overline{K}} \times B_{\overline{K}} \to \overline{K} \) is given by evaluating functions on elements: \( \langle f, \{t\} \rangle = f(t) \).

Let \( J \) be the augmentation ideal of \( B \). We again have a decomposition \( J = \bigoplus_{\mu} J_{\mu} \). For \( \mu \) non-trivial, the space \( J_{\mu} \otimes_R \overline{K} \) is spanned by the element

\[
e_{\mu} = \frac{1}{q-1} \sum_{t \in \mathbf{F}^\times} \mu^{-1}(t) \{t\}.
\]

The space \( J_1 \otimes_R \overline{K} \), on the other hand, is spanned by

\[
e_1 = -1 + \frac{1}{q-1} \sum_{t \in \mathbf{F}^\times} \{t\}.
\]

The vector spaces \( I_{\overline{K}} \) and \( J_{\overline{K}} \) are canonically dual. The bases \( \epsilon_{\mu} \) and \( e_{\mu} \) are dual bases, that is, \( \langle \epsilon_{\mu}, e_{\nu} \rangle = \delta_{\mu,\nu} \). We write \( \epsilon_i \) for \( e_{\chi_i} \).

Let \( Y_i = e_i^{-1} \epsilon_i \). Then \( Y_i \) is an element of \( J_{\chi_i} \). Of course, \( Y_i^p = \gamma_i Y_{i+1} \) for some \( \gamma_i \in R \). For \( \mu = \prod \chi_i^{\mu(i)} \), let \( Y_{\mu} = \prod Y_i^{\mu(i)} \), an element of \( J_{\mu} \). We also put \( e_{\mu} = \prod \epsilon_i^{\mu(i)} \), which is not equal to \( e_{\mu} \).

Let \( w_{\mu} = \langle X_{\mu}, Y_{\mu} \rangle \) and \( w_i = \langle X_i^p, Y_i^p \rangle \). The main work in understanding \( G \) is understanding the behavior of these numbers. Note that \( w_{\mu} = \langle \epsilon_{\mu}, e_{\mu} \rangle \) and \( w_i = \langle \chi_{i+1}, \epsilon_i^p \rangle \), so these numbers do not depend on \( G \). We will understand them by computing with a specific choice of \( G \).
3.3 Determining the w’s

In this section, we take $G$ to be the constant group scheme $\mathbf{F}$ over $R$. Thus $A$ is the ring of functions $\mathbf{F} \to R$, and $B$ is the group algebra $R[\mathbf{F}]$. The elements $\epsilon_\mu$ of $A_{\overline{\mathbf{F}}}$ belong to $A$, and the elements $e_\mu$ of $B_{\overline{\mathbf{F}}}$ belong to $B$.

Obviously, $e_\mu e_\nu = \epsilon_{\mu \nu}$. Since $\Delta(\epsilon_\mu)$ is a $\mu$-eigenvector of $\mathbf{F}^\times$, it must be a linear combination of the tensors $\epsilon_\mu \otimes 1, 1 \otimes \epsilon_\mu$, and the $e_\nu \otimes \epsilon_\eta$ with $\nu \eta = \mu$. As $\langle \Delta(\epsilon_\mu), \epsilon_\mu \otimes 1 \rangle = \langle \epsilon_\mu, \epsilon_\mu \rangle = 1$, it follows that $\epsilon_\mu \otimes 1$ appears with coefficient 1, and similarly for $1 \otimes \epsilon_\mu$. We thus find

$$\Delta(\epsilon_\mu) = \epsilon_\mu \otimes 1 + 1 \otimes \epsilon_\mu + \sum_{\nu \eta = \mu} J_{\nu, \eta} \cdot e_\nu \otimes \epsilon_\eta,$$

for some $J_{\nu, \eta} \in R$. Dualizing these expressions, we find $e_\mu e_\nu = J_{\mu, \nu} e_{\mu \nu}$ and

$$\Delta(e_\mu) = e_\mu \otimes 1 + 1 \otimes e_\mu + \sum_{\nu \eta = \mu} e_\nu \otimes e_\eta.$$

We now consider $B_k = k[\mathbf{F}]$. The space $I_k$ is spanned by the elements $\langle x \rangle = \{ x \} - \{ 0 \}$, while the space $I^2$ is spanned by the elements

$$\langle x \rangle \langle y \rangle = \{ x + y \} - \{ x \} - \{ y \} + \{ 0 \} = \langle x + y \rangle - \langle x \rangle - \langle y \rangle.$$

Thus $I_k/I^2_k$ is the quotient of the $k$-vector space with basis $\langle x \rangle$ (for $x \in \mathbf{F}$), by the relations $\langle x + y \rangle = \langle x \rangle + \langle y \rangle$. In other words, $I/I^2$ is exactly $k \otimes \mathbf{F}$. This decomposes as $\bigoplus_{i \in I} kv_i$, where the product is over the embeddings of $\mathbf{F}$ into $k$ (as fields). By definition, $\mathbf{F}^\times$ acts on $v_i$ through $\chi_i$. It follows that $v_i = e_i$ (or rather, its image in $B_k$).

Since the $e_i$ are elements of $I_k$ whose image in $I_k/I^2_k$ form a basis, it follows from Nakayama’s lemma that the $e_i$ generate $B_k$ as an algebra (and thus $B$ as well). Clearly, $P_k = 0$, and so the only non-zero monomials in the $e_i$ are the $e^\mu$, and so these form a $k$-basis for $I_k$. Thus the $e^\mu$ are an $R$-basis of $I$. In particular, $e_\mu$ and $e^\mu$ differ multiplicatively by a unit of $R$. (In fact, $e^\mu = w_\mu e_\mu$, by definition of $w_\mu$, so this shows that $w_\mu$ is a unit. We obtain a more precise statement below.)

If $x \in I$ then $x^p = 0$ modulo $p$. It follows that if $\mu(i) + \nu(i) \geq p$ for any $i$ (e.g., if $\mu \nu$ is a fundamental character) then $e^\mu e^\nu = 0$ mod $p$; thus $e_\mu e_\nu = 0$ mod $p$ as well, and so $J(\mu, \nu) = 0$ mod $p$ as well. In particular, we see that $\Delta(e_i) = e_i \otimes 1 + 1 \otimes e_i$ modulo $p$. Suppose $n = \sum \mu(i)$. Then

$$\langle e^\mu, e^\nu \rangle = \prod_{i \in I} e_i^{\mu(i)} \prod_{i \in I} e_i^{\nu(i)} = \langle \prod \Delta_n(e_i)^{\mu(i)} \otimes e_i^{\nu(i)} \rangle$$

where $\Delta_n: A \to A^\otimes n$ is repeated comultiplication (which is an algebra homomorphism). Considering this equation mod $p$, we can replace $\Delta_n(e_i)$ with $e_i \otimes 1 \otimes \cdots \otimes 1 + \cdots$, where the $\cdots$ are the symmetrical terms. The inner product is the coefficient of $\otimes e_i^{\nu(i)}$, which is easily seen to be $\prod \mu(i)!$. We have thus proven:

**Proposition 8.** We have $w_\mu = \prod \mu(i)!$ modulo $p$.

Similarly, we have

$$\langle e^\mu_i, e^\nu_i \rangle = \langle \Delta_p(e_i)^{\mu(i)} \otimes e_i^{\nu(i)} \rangle.$$

Now, we can write $\Delta_p(e_i)$ as $x + y$, where $x$ is a sum of things like $e_i \otimes 1 \cdots \otimes 1$, and $y$ is a multiple of $p$. We have $(x + y)^p = x^p + \cdots + y^p$, where the unwritten terms have both $y$’s and binomial coefficients, and are thus divisible by $p^2$; of course, $y^p$ is also divisible by $p^2$. We thus find $(x + y)^p = x^p$ mod $p^2$. Now, the coefficient of $e_i^{\nu(p)}$ in $x^p$ is $p!$, which is $-p$ modulo $p^2$. We have thus shown:

**Proposition 9.** $w_i = -p$ modulo $p^2$.  

3.4 Structure theorem

We now return to the general setting. The elements \(X^\mu\) span an \(R\)-submodule of \(I\), while the \(Y^\mu\) span an \(R\)-submodule of the dual module \(J\). The pairing \(\langle X^\mu, Y^\mu \rangle\) is a unit, namely \(w_i\) (and all other pairings vanish). It follows that the \(X^\mu\) span \(I\) and the \(Y^\mu\) span \(J\). Thus the \(X_i\) generate \(A\) as an algebra.

Recall \(X_i^p = \delta_i X_{i+1}\) and \(Y_i^p = \gamma_i Y_{i+1}\) and \(\delta_i \gamma_i = w_i\). As \(w_i = -p\) modulo \(p^2\), we see that \(v(\delta_i) \leq e\), where \(v\) is the valuation on \(K\) with \(v(p) = e\). We have thus shown:

**Theorem 10.** \(A\) is isomorphic to the quotient of \(R[X_i]\) by equations \(X_i^p = \delta_i X_{i+1}\), where \(\delta_i\) is an element of \(R\) of valuation at most \(e\).

3.5 Existence theorem

**Theorem 11.** Let \((\delta_i)_{i \in I}\) be elements of \(R\) of valuation at most \(e\), and let \(A\) be the quotient of \(R[X_i]\) by the equations \(X_i^p = \delta_i X_{i+1}\). Then there is a unique structure of a Raynaud \(F\)-module scheme on \(G = \text{Spec}(A)\) such that \([t]X_i = \chi_i(t)X_i\) for all \(t \in F^\times\).

**Proof.** Choose \(c_i \in \overline{K}^\times\) such that \(\delta_i = c_i^p/c_{i+1}\). Identify \(A[\overline{K}\) with the ring of functions on \(F\) via \(X_i = c_i \epsilon_i\). Then the monomials \(X^\mu\), and the unit 1, form an \(R\)-basis for \(A\).

Let \(B\) be the \(R\)-dual of \(A\), thought of as an \(R\)-submodule of \(\overline{K}[F]\). Since \(X^\mu = (\prod c_i^{\mu(i)}) \epsilon^\mu\), the dual basis \(Y^\mu\) is given by \(Y^\mu = (\prod c_i^{-\mu(i)}) \epsilon^\mu\). Let \(Y_i = c_i^{-1} \epsilon_i\). Then \(Y^\mu = (\prod c_i^{-\mu(i)}) \epsilon^\mu\), which differs by a unit from \(Y^\mu\). Thus the \(Y^\mu\) (and 1) span \(B\) as an \(R\)-module.

Of course, we still have \(Y_i^p = \gamma_i Y_{i+1}\) with \(\gamma_i \delta_i = p\). The point is that, due to the restriction on \(v(\delta_i)\), this shows that \(\gamma_i \in R\), and hence the span of the \(Y^\mu\) is an algebra. Thus the dual to \(A\) is closed under multiplication, which shows that \(A\) is closed under comultiplication, from which one can show that \(A\) is naturally a sub Hopf algebra of the ring of \(\overline{K}\)-valued functions on \(F\). This shows that \(G\) can be endowed with an \(F\)-module structure as stated.

By our previous work, any Raynaud \(F\)-module structure on \(G\) comes from one of the above form, for some choice of \(c\)'s. The choice of \(c\)'s does not change the resulting structure, as the \(c\)'s are uniquely determined up to a single \(q - 1\) root of unity, which can compensated for using \(F^\times\).

Thus the Raynaud \(F\)-module structure on \(G\) is unique. \(\square\)

We write \(G_\delta\) for the Raynaud \(F\)-module scheme corresponding to \(\delta = (\delta_i)_{i \in I}\). The structure theorem can be rephrased as: every Raynaud \(F\)-module scheme is isomorphic to \(G_\delta\), for some \(\delta\).

We leave the following proposition to the reader:

**Proposition 12.** The set of \(F\)-module homomorphism maps \(f: G_\delta \to G_{\delta'}\) correspond to sequence \((a_i)_{i \in I}\) of elements of \(R\) such that \(a_i + 1 \delta_i = a_i^p \delta'_i\). The sequence \((a_i)\) corresponds to the map \(f\) which is given on rings by \(X_i^p \mapsto a_i X_i\).

3.6 UP for \(F\)-module schemes

**Proposition 13.** Suppose \(e < p - 1\) and \(f: G \to G'\) is a map of Raynaud \(F\)-module schemes over \(R\) which induces an isomorphism over \(K\). Then \(f\) is an isomorphism.

**Proof.** Write \(G = G_\delta\) and \(G' = G_{\delta'}\), so that \(f\) corresponds to a sequence \((a_i)\) with \(a_i+1 \delta_i = a_i^p \delta'_i\). Let \(i\) be such that \(v(a_i)\) is maximal. Then \(v(a_i + \delta_i) \leq v(a_i) + e\), while \(v(a_i^p \delta'_i) \geq pe(a_i)\). Thus \(pe(a_i) \leq v(a_i) + e\), i.e., \((p-1)v(a_i) \leq e\). The hypothesis on \(e\) forces \(v(a_i) = 0\), and so all the \(a_i\)'s are units, and so \(f\) is an isomorphism. \(\square\)
Proposition 14. Suppose \( e < p - 1 \). Then UP holds for Raynaud \( \mathbf{F} \)-module schemes over \( K \).

Proof. Apply the previous proposition to the maximal and minimal prolongation, which are Raynaud \( \mathbf{F} \)-module schemes over \( R \). This shows \( G^+ = G^- \), which implies UP. 

This completes the proof of Raynaud’s theorem.