open.michigan

Author(s): Andrew Snowden

License: Unless otherwise noted, this material is made available under the terms of the **Creative Commons Attribution 3.0 License**: http://creativecommons.org/licenses/by/3.0/

We have reviewed this material in accordance with U.S. Copyright Law **and have tried to maximize your ability to use, share, and adapt it.** The citation key on the following slide provides information about how you may share and adapt this material.

Copyright holders of content included in this material should contact **open.michigan@umich.edu** with any questions, corrections, or clarification regarding the use of content.

For more information about **how to cite** these materials visit http://open.umich.edu/education/about/terms-of-use.

Any **medical information** in this material is intended to inform and educate and is **not a tool for self-diagnosis** or a replacement for medical evaluation, advice, diagnosis or treatment by a healthcare professional. Please speak to your physician if you have questions about your medical condition.

Viewer discretion is advised: Some medical content is graphic and may not be suitable for all viewers.

UNIVERSITY OF MICHIGAN

cc BY

Attribution Key

for more information see: http://open.umich.edu/wiki/AttributionPolicy

Use + Share + Adapt

{ Content the copyright holder, author, or law permits you to use, share and adapt. }

PD-GOV **Public Domain – Government**: Works that are produced by the U.S. Government. (17 USC § 105)

PD-EXP **Public Domain – Expired**: Works that are no longer protected due to an expired copyright term.

PD-SELF **Public Domain – Self Dedicated**: Works that a copyright holder has dedicated to the public domain.

(cc) ZERO **Creative Commons – Zero Waiver**

(cc) BY **Creative Commons – Attribution License**

(cc) BY-SA **Creative Commons – Attribution Share Alike License**

(cc) BY-NC **Creative Commons – Attribution Noncommercial License**

(cc) BY-NC-SA **Creative Commons – Attribution Noncommercial Share Alike License**

GNU-FDL **GNU – Free Documentation License**

Make Your Own Assessment

{ Content Open.Michigan believes can be used, shared, and adapted because it is ineligible for copyright. }

PD-INEL **Public Domain – Ineligible**: Works that are ineligible for copyright protection in the U.S. (17 USC § 102(b)) *laws in your jurisdiction may differ

{ Content Open.Michigan has used under a Fair Use determination. }

FAIR USE **Fair Use**: Use of works that is determined to be Fair consistent with the U.S. Copyright Act. (17 USC § 107) *laws in your jurisdiction may differ

Our determination **DOES NOT** mean that all uses of this 3rd-party content are Fair Uses and we **DO NOT** guarantee that your use of the content is Fair.

To use this content you should **do your own independent analysis** to determine whether or not your use will be Fair.

Lecture 7: Group schemes 3

The final lecture on group schemes is devoted entirely to Raynauds theorem: in mixed characteristic and low ramification, a group scheme is determined by its generic fiber. The proof proceeds by first reducing the statement to a special class of group schemes, the Raynaud F-module schemes. Next, these schemes are classified; this is the meat of the argument. (In lecture, I didnt have time to prove the important congruence relations, but they are covered in the notes here.) Finally, one verifies the theorem for Raynaud F-module schemes using the classification result.

A good reference for today is Tates article "Finite flat group schemes" in the book "Modular forms and Fermats last theorem" (MR1638478), especially section 4.

1 Finite flat group schemes

Up until now, we have considered finite commutative group schemes over a field. We now work over a more general base scheme S, which we assume to be affine and noetherian; write $S = \mathrm{Spec}(R)$. To have an analogous theory, we only consider flat group schemes. A finite flat R-module is projective, and so the coordinate rings of our group schemes will be projective R-modules.

Much of what we have previously done carries over:

- Finite flat group schemes over S correspond to Hopf algebras over R which are finitely generated and projective.
- We define the order of a finite flat group scheme as the rank of the corresponding Hopf algebra. In general, this is a locally constant function on S, but if S is connected it can be treated as a single number.
- Quotients work: if G and H are finite and flat and H is a closed subgroup of G then G/H exists and is finite and flat of the expected order.
- The classification of tale group schemes is similar: assuming S is connected, the category of finite flat commutative tale group schemes is equivalent to the category of finite $\pi_1(S, s)$-modules, where s is a geometric point of S. If R is the ring of integers in a finite extension K of $\mathbf{Q}_p$, then $\pi_1(S, s)$ is the Galois group of the maximal unramified extension of K.
- Assume R is henselian local (e.g., complete local). Then one can treat the connected–tale sequence in the same manner. If $G = \mathrm{Spec}(A)$ is a group scheme, then A is semi-local and thus decomposes as $A = \prod A_i$. Once again, $G^0 = \mathrm{Spec}(A_0)$, where the counit factors through A_0. One then has $G^{\mathrm{et}} = G/G^0$.
- If the order of G is invertible on S then G is tale.
- Cartier duality works in the same way.

These are notes for Math 679, taught in the Fall 2013 semester at the University of Michigan by Andrew Snowden.

2 Raynaud's theorem

2.1 Statement of theorem

Let $K/\mathbf{Q}_p$ be a finite extension, let R be its ring of integers in K, let k be the residue field of R, and let e be the ramification index of $K/\mathbf{Q}_p$. A prolongation of a finite group scheme G_0/K is a finite flat group scheme G/R equipped with an isomorphism $G_K \to G_0$.

Theorem 1 (Raynaud)**.** *Suppose $e < p-1$. Let G_0 be a finite commutative group scheme over K. Then any two prolongations of G_0 to R are isomorphic.*

Remark 2. This theorem is clearly not true without the hypothesis on e: indeed, if K has the pth roots of unity then μ_p and $\mathbf{Z}/p\mathbf{Z}$ are isomorphic over K, and thus both prolongations of $\mathbf{Z}/p\mathbf{Z}$, but are not isomorphic over R (as one is tale and one is connected). □

We say that a group scheme G_0 over K has property UP (unique prolongation) if any two prolongations of G_0 to R are isomorphic. Raynaud's theorem says UP holds for all G_0 if $e < p-1$.

The proof of Raynaud's theorem can be divided into three steps. First we show that if G_0 is an extension and UP holds for the sub and quotient then it holds for G_0. This means it suffices to prove UP for simple groups. We then classify the simple groups and their prolongations. This is the most involved step. Finally, we check by hand that UP holds when $e < p-1$.

2.2 Prolongations

Let $G_0 = \mathrm{Spec}(A_0)$ be a finite commutative group scheme over K. Prolongations of G_0 correspond to finite R-subalgebras A of A_0 which are closed under comultiplication and span A_0 over K (these conditions imply that A is closed under the antipode). We partially order prolongations using inclusion on rings.

Proposition 3. *Two prolongations have an inf and a sup.*

Proof. If A and A' are the rings of two prolongations then AA' is clearly closed under comultiplication, and thus a prolongation greater than each; it is clearly the unique minimal one, and thus the inf. Sups follow from Cartier duality. □

Proposition 4. *If G_0 has a prolongation then it has a maximal one G^+ and a minimal one G^-.*

Proof. Since A_0 is a finite tale K-algebra, it has a maximal order. It follows that any ascending chain of prolongations stabilizes, and so there exists a maximal prolongation. The minimal one follows from Cartier duality. □

We thus see that UP holds for G_0 if and only if the natural map $G^+ \to G^-$ is an isomorphism (or, in terms of rings, $A^+ = A^-$). In particular, one can check UP by passing to an extension of K.

Suppose now that

$$0 \to G_0' \to G_0 \to G_0'' \to 0$$

is a short exact sequence of group schemes and that G_0 admits a prolongation G. Then the scheme-theoretic closure G' of G_0' in G is a prolongation of G_0', and the quotient $G'' = G/G'$ is a prolongation of G_0''. Furthermore, if H is a second prolongation of G_0 which is less than G (i.e., there is a map $G \to H$), then clearly H' is less than G' and H'' is less than G'', and the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & G' \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H' \arrow[r] & H \arrow[r] & H'' \arrow[r] & 0
\end{tikzcd}$$

Thus, if the maps $G' \to H'$ and $G'' \to H''$ are isomorphisms, so is $G \to H$. This proves the following:

Proposition 5. *In the above situation, if G_0' and G_0'' satisfy UP then so does G_0.*

2.3 Raynaud F-module schemes

Let G_0/K be a simple group scheme of p-power order, and let $V = G_0(\overline{K})$, and irreducible $\mathbf{F}_p$-representation of the Galois group. Then $\mathbf{F} = \mathrm{End}_{G_K}(V)$ is a finite extension of $\mathbf{F}_p$, and we can regard V as an absolutely irreducible $\mathbf{F}$-linear representation of G_K.

Suppose now that k is algebraically closed, e.g., pass to K^{un}. Then G_K is an extension of a tame part I^t, which is abelian, by a wild part I^w, which is pro-p. Since I^w is pro-p, it must fix a non-zero vector in V; thus V^{I^w} is non-zero. But I^w is normal in G_K, and so V^{I^w} is a subrepresentation of V. Since V is simple, this implies $V = V^{I^w}$, i.e., I^w acts trivially. Thus the action of G_K on V factors through I^t. Since V (regarded as an $\mathbf{F}$-vector space) is an absolutely irreducible representation of an abelian group, it necessarily has dimension 1, i.e., $\dim_{\mathbf{F}}(V) = 1$.

An $\mathbf{F}$-module scheme (over R or K) is a group scheme G equipped with a ring homomorphism $\mathbf{F} \to \mathrm{End}(G)$. If $G = \mathrm{Spec}(A)$, we write $[t]$ for the map $A \to A$ induced by $t \in \mathbf{F}$. A Raynaud $\mathbf{F}$-module scheme G is an $\mathbf{F}$-module scheme of the same order as $\mathbf{F}$; thus it is an $\mathbf{F}$-module scheme such that $G(\overline{K})$ is one-dimensional over $\mathbf{F}$. The above discussion proves the following:

Proposition 6. *Suppose k is algebraically closed and G_0/K is simple of p-power order. Then G_0 is canonically a Raynaud $\mathbf{F}$-module scheme (for some $\mathbf{F}$).*

Proposition 7. *(No hypothesis on k.) Suppose UP holds for every Raynaud $\mathbf{F}$-module scheme over K^{un}. Then UP holds for all finite order group schemes over K.*

Proof. Let G_0/K be given. We can check UP for G_0 over K^{un}, so we can assume k is closed. We have shown that UP can be deduced if it is known for the outer groups in an extension. It thus suffices to treat the case where G_0 is simple. If G_0 is p-power order then it is a Raynaud $\mathbf{F}$-module scheme, and satisfies UP by hypothesis. If G_0 is prime-to-p then UP is automatic, as any prolongation is tale. □

Suppose now that G_0 is a Raynaud $\mathbf{F}$-module scheme over K. The action of $\mathbf{F}$ need not extend to an arbitrary prolongation of G_0. However, it necessarily extends to the maximal and minimal prolongations since they are unique. (For $t \in F^{\times}$, the map $[t]$ of A_0 is an automorphism of Hopf algebras, and therefore must carry A^+ into itself.) Thus G^+ and G^- are Raynaud $\mathbf{F}$-module schemes, and the map $G^+ \to G^-$ respects the $\mathbf{F}$-structure.

3 Analysis of Raynaud F-module schemes

3.1 Set-up

We now analyze Raynaud $\mathbf{F}$-module schemes. We fix the finite field $\mathbf{F}$, and write $q = p^r$ for its order. We assume that k contains the $q-1$ roots of unity (i.e., $\mathbf{F}$ embeds into k).

A character $\mathbf{F}^\times \to R^\times$ is fundamental if the composite map $\mathbf{F}^\times \to k^\times$ extends to an embedding of fields. If χ is a fundamental character, then any other one is of the form χ^{p^k} for some k. Enumerate the fundamental characters as $(\chi_i)_{i\in\mathcal{I}}$, for some index set $\mathcal{I}$. For $i \in I$, define $i+1$ by $\chi_{i+1} = \chi_i^p$. Then $\mathcal{I}$ is a torsor for $\mathbf{Z}/r\mathbf{Z}$.

An arbitrary character $\mu\colon \mathbf{F}^\times \to R^\times$ can be expressed as a product $\prod_{i\in I} \chi_i^{a_i}$ with $a_i \in \mathbf{Z}$. This product is unique if we impose the conditions that $0 \le a_i \le p-1$ and not all the a_i are equal to 0. We write $\mu(i)$ in place of a_i. Note that if μ is the trivial character then $\mu(i) = p-1$ for all i.

3.2 Initial analysis

Let $G = \mathrm{Spec}(A)$ be a Raynaud $\mathbf{F}$-module scheme over R. Then $G_{\overline{K}}$ is isomorphic to the constant group scheme $\mathbf{F}$ over $\overline{K}$. We fix an isomorphism of $A_{\overline{K}}$ with the algebra of functions $\mathbf{F} \to \overline{K}$. For a character $\mu\colon \mathbf{F}^\times \to K^\times$, we let ϵ_μ be the function $\mathbf{F} \to K$ extending μ and with $\epsilon_\mu(0) = 0$. Thus ϵ_μ is an element of $A_{\overline{K}}$. We put $\epsilon_i = \epsilon_{\chi_i}$.

Let I be the augmentation ideal of A. Since $\mathbf{F}^\times$ is a finite group whose order is invertible in R and all characters of $\mathbf{F}^\times$ are defined over R (since k contains the $q-1$ roots of unity), we can decompose I as a sum $\bigoplus_\mu I_\mu$, where the sum is over the characters μ of $\mathbf{F}^\times$, and I_μ is the R-submodule of I consisting of elements x such that $[t]x = \mu(t)x$ for all $t \in \mathbf{F}^\times$. Clearly, $I_\mu \otimes_R \overline{K}$ is spanned by ϵ_μ, and so I_μ is a free R-module of rank 1.

For each $i \in \mathcal{I}$, choose a non-zero element X_i of I_{χ_i}. Then $X_i = c_i\epsilon_i$ for some $c_i \in \overline{K}^\times$. Since X_i^p clearly belongs to $I_{\chi_{i+1}}$, we have $X_i^p = \delta_i X_{i+1}$ for some $\delta_i \in R$. As $\epsilon_i^p = \epsilon_{i+1}$, we find $\delta_i = c_i^p/c_{i+1}$. For a character $\mu = \prod \chi_i^{\mu(i)}$ of F, let X^μ be the product $\prod X_i^{\mu(i)}$, an element of I_μ. Note that what we might call ϵ^μ, namely $\prod \epsilon_i^{\mu(i)}$, is simply ϵ_μ.

Let $G^\vee = \mathrm{Spec}(B)$ be the Cartier dual of G, so that B is the R-linear dual of A. Then $B_{\overline{K}}$ is naturally identified with the group algebra $\overline{K}[\mathbf{F}]$. We write $\{t\}$ for the element of $\overline{K}[\mathbf{F}]$ corresponding to $t \in F$. Note that $\{t\}\{s\} = \{t+s\}$ (multiplication in the group algebra), but $[t]\{s\} = \{ts\}$ (the $\mathbf{F}$-module structure). Note also that the pairing $\langle,\rangle\colon A_{\overline{K}} \times B_{\overline{K}} \to \overline{K}$ is given by evaluating functions on elements: $\langle f, \{t\}\rangle = f(t)$.

Let J be the augmentation ideal of B. We again have a decomposition $J = \bigoplus_\mu J_\mu$. For μ non-trivial, the space $J_\mu \otimes_R \overline{K}$ is spanned by the element

$$e_\mu = \frac{1}{q-1} \sum_{t\in F^\times} \mu^{-1}(t)\{t\}.$$

The space $J_1 \otimes_R \overline{K}$, on the other hand, is spanned by

$$e_1 = -1 + \frac{1}{q-1} \sum_{t\in F^\times} \{t\}.$$

The vector spaces $I_{\overline{K}}$ and $J_{\overline{K}}$ are canonically dual. The bases ϵ_μ and e_μ are dual bases, that is, $\langle \epsilon_\mu, e_\nu\rangle = \delta_{\mu,\nu}$. We write e_i for e_{χ_i}.

Let $Y_i = c_i^{-1}e_i$. Then Y_i is an element of J_{χ_i}. Of course, $Y_i^p = \gamma_i Y_{i+1}$ for some $\gamma_i \in R$. For $\mu = \prod \chi_i^{\mu(i)}$, let $Y^\mu = \prod Y_i^{\mu(i)}$, an element of J_μ. We also put $e^\mu = \prod e_i^{\mu(i)}$, which is not equal to e_μ.

Let $w_\mu = \langle X^\mu, Y^\mu\rangle$ and $w_i = \langle X_i^p, Y_i^p\rangle$. The main work in understanding G is understanding the behavior of these numbers. Note that $w_\mu = \langle \epsilon_\mu, e^\mu\rangle$ and $w_i = \langle \chi_{i+1}, e_{\chi_i}^p\rangle$, so these numbers do not depend on G. We will understand them by computing with a specific choice of G.

3.3 Determining the w's

In this section, we take G to be the constant group scheme $\mathbf{F}$ over R. Thus A is the ring of functions $\mathbf{F} \to R$, and B is the group algebra $R[\mathbf{F}]$. The elements ϵ_μ of $A_{\overline{K}}$ belong to A, and the elements e_μ of $B_{\overline{K}}$ belong to B.

Obviously, $\epsilon_\mu \epsilon_\nu = \epsilon_{\mu\nu}$. Since $\Delta(\epsilon_\mu)$ is a μ-eigenvector of $\mathbf{F}^\times$, it must be a linear combination of the tensors $\epsilon_\mu \otimes 1$, $1 \otimes \epsilon_\mu$, and the $\epsilon_\nu \otimes \epsilon_\eta$ with $\nu\eta = \mu$. As $\langle \Delta(\epsilon_\mu), e_\mu \otimes 1 \rangle = \langle \epsilon_\mu, e_\mu \rangle = 1$, it follows that $\epsilon_\mu \otimes 1$ appears with coefficient 1, and similarly for $1 \otimes \epsilon_\mu$. We thus find

$$\Delta(\epsilon_\mu) = \epsilon_\mu \otimes 1 + 1 \otimes \epsilon_\mu + \sum_{\nu\eta=\mu} J_{\nu,\eta} \cdot \epsilon_\nu \otimes \epsilon_\eta,$$

for some $J_{\nu,\eta} \in R$. Dualizing these expressions, we find $e_\mu e_\nu = J_{\mu,\nu} e_{\mu\nu}$ and

$$\Delta(e_\mu) = e_\mu \otimes 1 + 1 \otimes e_\mu + \sum_{\mu=\nu\eta} e_\nu \otimes e_\eta.$$

We now consider $B_k = k[\mathbf{F}]$. The space I_k is spanned by the elements $\langle x \rangle = \{x\} - \{0\}$, while the space I^2 is spanned by the elements

$$\langle x \rangle \langle y \rangle = \{x + y\} - \{x\} - \{y\} + \{0\} = \langle x + y \rangle - \langle x \rangle - \langle y \rangle.$$

Thus I_k/I_k^2 is the quotient of the k-vector space with basis $\langle x \rangle$ (for $x \in \mathbf{F}$), by the relations $\langle x + y \rangle = \langle x \rangle + \langle y \rangle$. In other words, I/I^2 is exactly $k \otimes \mathbf{F}$. This decomposes as $\bigoplus_{i \in \mathcal{I}} kv_i$, where the product is over the embeddings of $\mathbf{F}$ into k (as fields). By definition, $\mathbf{F}^\times$ acts on v_i through χ_i. It follows that $v_i = e_i$ (or rather, its image in B_k).

Since the e_i are elements of I_k whose image in I_k/I_k^2 form a basis, it follows from Nakayama's lemma that the e_i generate B_k as an algebra (and thus B as well). Clearly, $I_k^p = 0$, and so the only non-zero monomials in the e_i are the e^μ, and so these form a k-basis for I_k. Thus the e^μ are an R-basis of I. In particular, e_μ and e^μ differ multiplicatively by a unit of R. (In fact, $e^\mu = w_\mu e_\mu$, by definition of w_μ, so this shows that w_μ is a unit. We obtain a more precise statement below.)

If $x \in I$ then $x^p = 0$ modulo p. It follows that if $\mu(i) + \nu(i) \geq p$ for any i (e.g., if $\mu\nu$ is a fundamental character) then $e^\mu e^\nu = 0 \bmod p$; thus $e_\mu e_\nu = 0 \bmod p$ as well, and so $J_{\mu,\nu} = 0 \bmod p$ as well. In particular, we see that $\Delta(\epsilon_i) = \epsilon_i \otimes 1 + 1 \otimes \epsilon_i$ modulo p. Suppose $n = \sum \mu(i)$. Then

$$\langle \epsilon^\mu, e^\mu \rangle = \langle \prod \epsilon_i^{\mu(i)}, \prod e_i^{\mu(i)} \rangle = \langle \prod \Delta_n(\epsilon_i)^{\mu(i)}, \bigotimes e_i^{\otimes \mu(i)} \rangle$$

where $\Delta_n \colon A \to A^{\otimes n}$ is repeated comultiplication (which is an algebra homomorphism). Considering this equation mod p, we can replace $\Delta_n(\epsilon_i)$ with $\epsilon_i \otimes 1 \otimes \cdots \otimes 1 + \cdots$, where the $\cdots$ are the symmetrical terms. The inner product is the coefficient of $\bigotimes \epsilon_i^{\otimes \mu(i)}$, which is easily seen to be $\prod \mu(i)!$. We have thus proven:

Proposition 8. *We have $w_\mu = \prod \mu(i)!$ modulo p.*

Similarly, we have

$$\langle \epsilon_i^p, e_i^p \rangle = \langle \Delta_p(\epsilon_i)^p, e_i^{\otimes p} \rangle.$$

Now, we can write $\Delta_p(\epsilon_i)$ as $x + y$, where x is a sum of things like $\epsilon_i \otimes 1 \cdots \otimes 1$, and y is a multiple of p. We have $(x + y)^p = x^p + \cdots + y^p$, where the unwritten terms have both y's and binomial coefficients, and are thus divisible by p^2; of course, y^p is also divisible by p^2. We thus find $(x + y)^p = x^p \bmod p^2$. Now, the coefficient of $\epsilon_i^{\otimes p}$ in x^p is $p!$, which is $-p$ modulo p^2. We have thus shown:

Proposition 9. *$w_i = -p$ modulo p^2.*

3.4 Structure theorem

We now return to the general setting. The elements X^μ span an R-submodule of I, while the Y^μ span an R-submodule of the dual module J. The pairing $\langle X^\mu, Y^\mu \rangle$ is a unit, namely w_χ (and all other pairings vanish). It follows that the X^μ span I and the Y^μ span J. Thus the X_i generate A as an algebra.

Recall $X_i^p = \delta_i X_{i+1}$ and $Y_i^p = \gamma_i Y_{i+1}$ and $\delta_i \gamma_i = w_i$. As $w_i = -p$ modulo p^2, we see that $v(\delta_i) \le e$, where v is the valuation on K with $v(p) = e$. We have thus shown:

Theorem 10. *A is isomorphic to the quotient of $R[X_i]$ by equations $X_i^p = \delta_i X_{i+1}$, where δ_i is an element of R of valuation at most e.*

3.5 Existence theorem

Theorem 11. *Let $(\delta_i)_{i \in \mathcal{I}}$ be elements of R of valuation at most e, and let A be the quotient of $R[X_i]$ by the equations $X_i^p = \delta_i X_{i+1}$. Then there is a unique structure of a Raynaud F-module scheme on $G = \operatorname{Spec}(A)$ such that $[t]X_i = \chi_i(t) X_i$ for all $t \in F^\times$.*

Proof. Choose $c_i \in \overline{K}^\times$ such that $\delta_i = c_i^p / c_{i+1}$. Identify $A_{\overline{K}}$ with the ring of functions on $\mathbf{F}$ via $X_i = c_i \epsilon_i$. Then the monomials X^μ, and the unit 1, form an R-basis for A.

Let B be the R-dual of A, thought of as an R-submodule of $\overline{K}[\mathbf{F}]$. Since $X^\mu = (\prod c_i^{\mu(i)}) \epsilon_\mu$, the dual basis Y_μ is given by $Y_\mu = (\prod c_i^{-\mu(i)}) e_\mu$. Let $Y_i = c_i^{-1} e_i$. Then $Y^\mu = (\prod c_i^{-\mu(i)}) e^\mu$, which differs by a unit from Y_μ. Thus the Y^μ (and 1) span B as an R-module.

Of course, we still have $Y_i^p = \gamma_i Y_{i+1}$ with $\gamma_i \delta_i = p$. The point is that, due to the restriction on $v(\delta_i)$, this shows that $\gamma_i \in R$, and hence the span of the Y^μ is an algebra. Thus the dual to A is closed under multiplication, which shows that A is closed under comultiplication, from which one can show that A is naturally a sub Hopf algebra of the ring of $\overline{K}$-valued functions on $\mathbf{F}$. This shows that G can be endowed with an $\mathbf{F}$-module structure as stated.

By our previous work, any Raynaud $\mathbf{F}$-module structure on G comes from one of the above form, for some choice of c's. The choice of c's does not change the resulting structure, as the c's are uniquely determined up to a single $q - 1$ root of unity, which can compensated for using $\mathbf{F}^\times$. Thus the Raynaud $\mathbf{F}$-module structure on G is unique. □

We write G_δ for the Raynaud $\mathbf{F}$-module scheme corresponding to $\delta = (\delta_i)_{i \in \mathcal{I}}$. The structure theorem can be rephrased as: every Raynaud $\mathbf{F}$-module scheme is isomorphic to G_δ, for some δ. We leave the following proposition to the reader:

Proposition 12. *The set of $\mathbf{F}$-module homomorphism maps $f \colon G_\delta \to G_{\delta'}$ correspond to sequence $(a_i)_{i \in \mathcal{I}}$ of elements of R such that $a_{i+1} \delta_i = a_i^p \delta_i'$. The sequence (a_i) corresponds to the map f which is given on rings by $X_i' \mapsto a_i X_i$.*

3.6 UP for F-module schemes

Proposition 13. *Suppose $e < p - 1$ and $f \colon G \to G'$ is a map of Raynaud $\mathbf{F}$-module schemes over R which induces an isomorphism over K. Then f is an isomorphism.*

Proof. Write $G = G_\delta$ and $G' = G_{\delta'}$, so that f corresponds to a sequence (a_i) with $a_{i+1} \delta_i = a_i^p \delta_i'$. Let i be such that $v(a_i)$ is maximal. Then $v(a_{i+1} + \delta_i) \le v(a_i) + e$, while $v(a_i^p \delta_i') \ge p v(a_i)$. Thus $p v(a_i) \le v(a_i) + e$, i.e., $(p - 1) v(a_i) \le e$. The hypothesis on e forces $v(a_i) = 0$, and so all the a_j's are units, and so f is an isomorphism. □

Proposition 14. *Suppose $e < p - 1$. Then UP holds for Raynaud* $\mathbf{F}$*-module schemes over K.*

Proof. Apply the previous proposition to the maximal and minimal prolongation, which are Raynaud $\mathbf{F}$-module schemes over R. This shows $G^+ = G^-$, which implies UP. □

This completes the proof of Raynaud's theorem.