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Lecture 9: Néron models

This lecture is an exposition on Néron models. I begin by discussing quasi-finite étale group schemes over DVRs: these are the sorts of things that occur as the prime-to-p torsion of Néron models. Then, before going to the general theory, I discuss Néron models of elliptic curves, especially their relationship to Weierstrass and minimal regular models. I do two simple and explicit examples. Finally, I introduce Néron models of general abelian varieties. Two applications are discussed: the NéronOggShafarevich criterion, and Grothendiecks generalization thereof; and the semi-stable reduction theorem.

1 Quasi-finite étale group schemes

At the end of the last lecture, we considered the group scheme obtained by taking the n torsion in the smooth locus of a minimal Weierstrass model for an elliptic curve. This group scheme is typically not finite over the base. However, it is quasi-finite: all its fibers have finitely many points. We now study such group schemes in the étale case.

Let R be a henselian DVR, and keep our usual notation (K, k, etc). Let G be a quasi-finite étale group scheme over R (assumed to be of finite presentation and commutative). Let $M = G(\overline{K})$ be the Galois module corresponding to G_K, and let $M_0 = G(\overline{k})$ be the one corresponding to G_k. Since G is étale, the natural map $G(\overline{R}) \to G(\overline{k})$ is an isomorphism, and so we can regard M_0 as a submodule of M. It is obviously stable under the Galois action and fixed by inertia.

Theorem 1. *The functor $G \mapsto (M, M_0)$ is an equivalence of categories.*

Some comments:

- Let G correspond to (M, M_0) and let H be a subgroup corresponding to (N, N_0). Then H is closed in G if and only if $N_0 = M_0 \cap N$. In this case, G/H is an étale quasi-finite group scheme, and it corresponds to $(M/N, M_0/N_0)$.

- Let $\mathcal{G}$ be a finite group scheme over K, corresponding to the Galois module M. Then $\mathcal{G}$ admits a maximal extension to an étale quasi-finite group over R, by taking $M_0 = M^{I_K}$. It also admits a minimal such extension, by taking $M_0 = 0$; we call this the extension by zero.

- Let G correspond to (M, M_0), and let H be the closed subgroup corresponding to (M_0, M_0). Then H is the maximal closed subgroup of G which is finite over R. Note that $H_k = G_k$.

- Suppose G is a quasi-finite flat group scheme over R which is killed by n, and n is invertible on R. Then G_K and G_k are both étale, and this implies that G itself is étale. In particular, if $\mathcal{E}$ is some smooth commutative group variety over R and $\mathcal{E}[n]$ is quasi-finite, then it is étale as well.

- Remark. At the end of the previous lecture, we proved that if E is an elliptic curve with semi-stable reduction, then I_K fixes a vector in $T_\ell(E)$. Let us re-explain the argument with the above theory in hand. Let G be the ℓ^n-torsion in the smooth part of the minimal Weiestrass model. Then this is a quasi-finite étale group scheme over R. Let $H \subset G$ be the maximal finite subgroup. Then $H_k = G_k$, and so $H(\overline{k})$ contains a point of order ℓ^n (since it is the ℓ^n-torsion in either an elliptic curve or $\mathbf{G}_m$). Since H is étale over R, the map $H(K^{\mathrm{un}}) \to H(\overline{k})$

These are notes for Math 679, taught in the Fall 2013 semester at the University of Michigan by Andrew Snowden.

is an isomorphism. Of course, $H(K^{\mathrm{un}}) \subset E[\ell^n](K^{\mathrm{un}})$, so this shows that E contains a point of order ℓ^n defined over K^{un}.

2 Néron models of elliptic curves

A good reference for this section is Chapter IV of Silverman's book "Advanced topics in the arithmetic of elliptic curves" (MR1312368).

2.1 Motiviation

Let E/K be an elliptic curve and let $\mathcal{W}/R$ be its minimal Weierstrass model. Since $\mathcal{W}$ is proper over R, we have $\mathcal{W}(R) = \mathcal{W}(K) = E(K)$. However, $\mathcal{W}$ is typically singular. Its smooth locus $\mathcal{W}_{\mathrm{sm}}$ is a group scheme over R. Typically, it is not proper, and not all K-points of E extend to $\mathcal{W}_{\mathrm{sm}}$. Those that do are the subgroup $E_0(K)$, which has finite index in $E(K)$.

The Néron model is an extension $\mathcal{E}$ of E over R which combines the desirable properties of $\mathcal{W}$ and $\mathcal{W}_{\mathrm{sm}}$: it is a smooth group scheme and all K-points extend to R-points. The identity component of $\mathcal{E}$ is $\mathcal{W}_{\mathrm{sm}}$, while the component group of $\mathcal{E}_k$ (at least for k algebraically closed) is $E(K)/E_0(K)$. So all the points of $E(K)$ extend to points of $\mathcal{E}(R)$, and $E_0(K)$ is the subgroup of points which extend to the identity component of $\mathcal{E}$.

2.2 Minimal regular models and Neron models

Let C/K be a curve. A regular model for C is a proper flat scheme $\mathcal{C}$ over R which is regular and whose generic fiber is C. A regular model $\mathcal{C}$ is minimal if for any other regular model $\mathcal{C}'$, there exists a map of schemes $\mathcal{C}' \to \mathcal{C}$ extending the identity on the generic fiber. The main theorem is that minimal regular models exist and are canonically unique. One can find a regular model for C by starting with any model and repeatedly blowing-up and normalizing. From there, one can find a minimal regular model by blowing-down certain divisors in the special fiber.

Let E/K be an elliptic curve and let $\mathcal{C}/R$ be its minimal regular model. The Néron model of E is then the smooth locus in $\mathcal{C}$. (This can be taken as a definition, though a better definition is given below.)

2.3 Example 1

Consider the curve $y^2 = x^3 + p$ over $K = \mathbf{Q}_p$. The same equation defines the minimal Weierstrass model $\mathcal{W}$ over $R = \mathbf{Z}_p$. Clearly, $\mathcal{W}$ is smooth everywhere except for the point $P = (0, 0)$ in the special fiber.

We claim that P is regular. To see this, let $A = R[x, y]/(y^2 = x^3 + p)$ be the ring of natural affine chart containing P, so that P corresponds to the maximal ideal $\mathfrak{m} = (x, y, p)$. The ideal $\mathfrak{m}^2$ is generated by $x^2, xy, y^2, px, py, p^2$. But note that $y^2 = x^3 + p$, and $x^3 \in \mathfrak{m}$, so we may as well replace the generator y^2 with p, which means the generators px, py, and p^2 are unnecessary. Thus $\mathfrak{m}^2 = (x^2, xy, p)$. The quotient $\mathfrak{m}/\mathfrak{m}^2$ is has for a basis the images of x and y, and is thus two dimensional over the residue field $A/\mathfrak{m}$. Since A has Krull dimension 2, this establishes regularity.

It follows that $\mathcal{W}$ is a regular model for E, which is necessarily minimal since there are no divisors in the special fiber to blow-down. The Néron model $\mathcal{E}$ is the smooth locus of $\mathcal{W}$, i.e., $\mathcal{W} \setminus \{P\}$. In particular, the special fiber $\mathcal{E}_k$ is connected and isomorphic to $\mathbf{G}_a$. We have $E(K) = E_0(K)$ in this case.

2.4 Example 2

Now consider the curve E defined by $y^2 = x^3 + p^2$. Again, this equation defines the minimal Weierstrass model $\mathcal{W}$ over R and $P = (0, 0)$ in the special fiber is the unique singular point.

In this case, P is not regular. Let $A = R[x, y]/(y^2 = x^3 + p^2)$ and $\mathfrak{m} = (x, y, p)$, similar to before. The generators of $\mathfrak{m}^2$ are similar to before. The difference is that one can no longer use the defining equation to find p in $\mathfrak{m}^2$; in fact, the equation shows that y^2 is not needed as a generator of $\mathfrak{m}^2$. Thus $\mathfrak{m}^2 = (x^2, xy, px, py, p^2)$. The images of x, y, and p in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent (and in fact a basis), so $\mathfrak{m}/\mathfrak{m}^2$ is 3 dimensional, and so P is not regular.

To find the minimal regular model of E, we blow up at the point P. We'll only do the computations in the affine chart $\mathrm{Spec}(A)$. The blow-up algebra B is the subring of $A[t]$ generated by tx, ty, and tp. This is considered as a graded ring by giving t degree 1. The blow-up is $\mathrm{Proj}(B)$. Let B_1 be the degree 0 subring of $B[1/tx]$, and define B_2 and B_2 similarly but with ty and tp. Let $U_i = \mathrm{Spec}(B_i)$. Then $\mathrm{Proj}(B)$ is covered by the U_i, so we first study them.

The ring B_1 can be presented as the quotient of $R[x, y/x, p/x]$ by the equations $x(p/x) = p$ and $(y/x)^2 = x + (p/x)^2$. (One should think of y/x and p/x as indeterminates.) The special fiber is therefore defined by the equations $x(p/x) = 0$ and $(y/x)^2 = x + (p/x)^2$. This is a union of three lines: when $x = 0$ we get $(y/x) = \pm(p/x)$ and when $(p/x) = 0$ we get $x = (y/x)^2$. The three lines intersect at the point $x = (p/x) = (y/x) = 0$.

The ring B_2 can be presented as the quotient of $R[y, x/y, p/y]$ by the equations $y(p/y) = p$ and $1 = y(x/y)^3 + (p/y)^2$. Its special fiber is defined by $y(p/y) = 0$ and $1 = y(x/y)^3 + (p/y)^2$. This is also a union of three lines: when $y = 0$ we get $(p/y) = \pm 1$ and when $(p/y) = 0$ we get $y = (x/y)^{-3}$. Note that these lines do not intersect, since (p/y) is constant on each line of a different value. The two lines with $y = 0$ meet up with the two lines in U_1 with $x = 0$. Since y/x can assume any non-zero value in U_1 and x/y can assume any non-zero value in U_2, they glue to $\mathbf{P}^1$'s. The third line in U_2 is missing two points, and is contained in the third line in U_1.

Finally, the ring B_3 can be presented as the quotient of $R[x/p, y/p]$ by the equation $(y/p)^2 = p(x/p)^3 + 1$. Its special fiber consists of two lines, defines by $(x/p) = 0$ and $(y/p) = \pm 1$. Thus U_3 is contained in $U_1 \cup U_2$.

We thus see that the special fiber of $\mathrm{Proj}(B)$ has three components, two $\mathbf{P}^1$'s and one $\mathbf{A}^1$, and they are joined at a single point. However, $\mathrm{Proj}(B)$ is not the full blow-up of $\mathcal{W}$ at P, but only one chart. The other chart adds the missing point to the $\mathbf{A}^1$ in the special fiber.

This blow-up $\mathcal{C}$ is the minimal regular model for E. The Néron model $\mathcal{E}$ is obtained by deleting the intersection point in the special fiber. Thus $\mathcal{E}^\circ$ has three components, so its component group is necessarily $\mathbf{Z}/3\mathbf{Z}$.

2.5 Classification of minimal regular models

If E has good reduction, then its minimal Weierstrass model is smooth, and coincides with its minimal regular model and Néron model. In this case, the special fiber of the minimal regular model is an elliptic curve.

In all other cases, the special fiber of the minimal regular model is made up of genus 0 curves, though they may have singularities and non-reduced behavior. This data is combinatorial, since one simply needs to record how many $\mathbf{P}^1$'s there are, how they intersect, and what their multiplicities are. It can be depicted as a sort of graph, with numbers on the edges to denote multiplicities. Néron and Kodiara classified all the possible special fibers; the graphs that occur turn out to be closely related to Dynkin diagrams. An important fact that follows from this classification is that, unless E has split multiplicative reduction, the special fiber of its Neron model has at most 4 components.

3 Néron models for abelian varieties

3.1 Definition and basic properties

It is not at all clear how to extend our discussion of elliptic curves to higher dimensional abelian varieties: the theory of Weierstrass models relies on explicit equations, which are unavailable, while the more abstract theory of minimal regular models is specific to curves. The key observation is that the functor of points of the Néron model of an elliptic curve admits a nice description.

Theorem 2. *Let E/K be an elliptic curve, and let $\mathcal{E}/R$ be its Néron model. Let $\mathcal{X}/R$ be any smooth scheme, and let $X = \mathcal{X}_K$. Then the natural map $\mathrm{Hom}_R(\mathcal{X}, \mathcal{E}) \to \mathrm{Hom}_K(X, E)$ is an isomorphism.*

Given this description of $\mathcal{E}$, it is clear how the definition can be extended to any scheme:

Definition 3. Let A/K be a smooth scheme. A Néron model for A is a smooth scheme $\mathcal{A}/R$ which satisfies the Néron mapping property: the natural map $\mathrm{Hom}_R(\mathcal{X}, \mathcal{A}) \to \mathrm{Hom}_K(X, A)$ is a bijection, for any smooth scheme $\mathcal{X}/R$ as above. □

Some remarks:

- The definition of Néron model specifies its functor of points on smooth R-schemes. Since the Néron model itself is required to be a smooth R-scheme, Yoneda's lemma shows that Néron models are canonically unique, when they exist.
- Although the definition applies to any smooth scheme A/K, we only consider the case where A is an abelian variety.
- The main existence result is that the Néron model of an abelian variety exists.
- As a special case of the Néron mapping property, we see that the natural map $\mathcal{A}(R) \to A(K)$ is a bijection, i.e., all K-points of A extend to R-points of $\mathcal{A}$. Thus, from the perspective of K-points, the Néron model behaves as if it were proper. This is not true for K'-points if K'/K is a ramified extension!
- Formation of Néron models is compatible with passing to unramified extensions, but not to ramified extensions, in general. Precisely, suppose K'/K is a finite extension, let $\mathcal{A}$ be the Néron model of A and let $\mathcal{A}'$ be the Néron model of $A_{K'}$. Then there is a natural map $\mathcal{A}_{R'} \to \mathcal{A}'$. If K'/K is unramified this map is an isomorphism, but when K'/K is ramified it is typically not. In particular, the natural map $\mathcal{A}(R') \to A(K')$ need not be an isomorphism.

3.2 Types of reduction

Let A/K be an abelian variety with Néron model $\mathcal{A}$, and let $\mathcal{A}_0$ be the special fiber of $\mathcal{A}$. Let $\mathcal{A}_0^\circ$ be its identity component. A theorem of Chevalley states that every smooth connected group is an extension of an abelian variety by a smooth affine group. Thus there is an exact sequence

$$0 \to L \to \mathcal{A}_0^\circ \to B \to 0,$$

where B is an abelian variety and L is a commutative smooth affine group. The group L contains a maximal torus T such that the quotient $U = L/T$ is unipotent (a product of $\mathbf{G}_a$'s). In other words, we can say that there is a canonical filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 = \mathcal{A}_0$$

where $T = F_1/F_0$ is a torus, $U = F_2/F_1$ is unipotent, $B = F_3/F_2$ is an abelian variety, and F_4/F_3 is finite étale (the component group). The dimensions of T, U, and B are important invariants of A refining the trichotomy of multiplicative/additive/good reduction in the case of elliptic curves.

We say that A has good reduction if it extends to an abelian scheme over R. (An abelian scheme is a smooth proper group scheme with geometrically connected fibers.) This is equivalent to $\mathcal{A}_0$ (or just $\mathcal{A}_0^\circ$) being an abelian variety. If A has good reduction then $\mathcal{A}$ is the unique abelian scheme extending it.

We say that A has semi-stable reduction if $\mathcal{A}_0$ has no unipotent part, i.e., $\mathcal{A}_0^\circ$ is an extension of an abelian variety by a torus (what is called a semi-abelian variety).

3.3 Neron–Ogg–Shafarevich

Let ℓ be a prime different from the residue characteristic and $T_\ell(A)$ the ℓ-adic Tate module of A, a representation of G_K.

Theorem 4. *A has good reduction if and only if $T_\ell(A)$ is unramified.*

Proof. The proof is the same as the elliptic curve case. We briefly recall the details. First, if A has good reduction then it extends to an abelian scheme $\mathcal{A}$, so $\mathcal{A}[\ell^n]$ is a finite étale group scheme over R, which implies that $T_\ell(A)$ is unramified. Conversely, suppose $T_\ell(A)$ is unramified. Then all the ℓ^n-torsion of A is defined over K^{un}, and so $\mathcal{A}(K^{\mathrm{un}})[\ell^n]$ has cardinality ℓ^{2ng}, where $g = \dim(A)$. Since the reduction map $A(K^{\mathrm{un}})[\ell^n] \to \mathcal{A}(\overline{k})$ is injective, we see that $\mathcal{A}(\overline{k})[\ell^n]$ has cardinality at least ℓ^{2ng}. Using the fact that $\mathbf{G}_m$ and $\mathbf{G}_a$ have too little ℓ^n-torsion, this implies that there can be no toric or unipotent part of $\mathcal{A}_0$, and so $\mathcal{A}_0^\circ$ is an abelian variety, which implies good reduction. □

There is an important generalization of this theorem, due to Grothendieck. We do not give the proof.

Theorem 5. *A has semi-stable reduction if and only if the action of I_K on $T_\ell(A)$ is unipotent.*

3.4 Semi-stable reduction theorem

Theorem 6. *There exists a finite extension K'/K such that $A_{K'}$ has semi-stable reduction.*

Proof. We assume K is a finite extension of $\mathbf{Q}_p$ for simplicity. By Grothendieck's extension of Néron–Ogg–Shafarevich, it is enough to show that $I_{K'}$ acts unipotently on $T_\ell(A)$ for some finite K'/K. In fact, we will show that this is true for any ℓ-adic representation of G_K!

Thus let V be a continuous ℓ-adic representation of G_K. Since the wild inertia subgroup of G_K is pro-p, its image in $\mathbf{GL}(V)$ must be finite. Thus, passing to a finite extension, we can assume wild inertia acts trivially. The quotient of G_K by the wild inertia subgroup is topologically generated by two elements, F (a lift of Frobenius) and τ (a generator of tame inertia), which satisfy the single relation $F\tau F^{-1} = \tau^q$. This equation shows that the transformations τ and τ^q of V are conjugate. Thus if $\alpha_1, \ldots, \alpha_n$ are the eigenvalues of τ then $\alpha_i^q = \alpha_{\sigma(i)}$ for some permutation $\sigma \in S_n$. This implies that $\alpha_i^{q^n} = \alpha_i$ for all i, i.e., the α_i are roots of unity of order dividing $q^n - 1$. Thus, by passing to an extension of K with ramification index $e = q^n - 1$ (which has the effect of replacing τ by τ^e), the action of τ (and thus all of inertia) becomes unipotent. □