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Lecture 10: Jacobians

This lecture is an exposition on Jacobians. I start with the analytic theory, where the Jacobian is defined as the quotient of a vector space by a lattice. I then discuss the representability issues of the functor of points over arbitrary fields. Following this, I briefly sketch Weil's construction of the Jacobian and say something about the relative situation.

1 Analytic theory

1.1 Hodge theory of curves

Let X be a (connected, smooth, projective) curve over the complex numbers of genus g. Let V be the space of global holomorphic 1-forms on X, which is a complex vector space of dimension g. Let $\mathrm{H}^1_{\mathrm{dR}}(X)$ be the first de Rham cohomology group of X, i.e., the space of smooth closed real 1-forms modulo exact forms. Since every element of V is closed, we have a natural map $V \to \mathrm{H}^1_{\mathrm{dR}}(X) \otimes_{\mathbf{R}} \mathbf{C}$.

Lemma 1. *This map is injective.*

Proof. Suppose $\omega \in V$ is closed, and write $\omega = df$. Then f is holomorphic: indeed, in local coordinates, this expression implies that df has no $d\overline{z}$, which is exactly the Cauchy–Riemann equations. Thus f is a holomorphic function on all of X, and therefore constant, and so $\omega = 0$. □

Theorem 2 (Hodge decomposition)**.** *The map $V \oplus \overline{V} \to \mathrm{H}^1_{\mathrm{dR}}(X) \otimes_{\mathbf{R}} \mathbf{C}$ is an isomorphism.*

Proof. Let $J \colon T_x \to T_x$ be the multiplication by i map on tangent spaces. For a complex 1-form ω on X, define ω^c by $-i\omega J$. Then $(-)^c$ induces an involution of $\mathrm{H}^1_{\mathrm{dR}}(X) \otimes_{\mathbf{R}} \mathbf{C}$. Clearly, V lies in the 1 eigenspace of this operator, while $\overline{V}$ lies in the -1 eigenspace. Thus $V \cap \overline{V} = 0$ in $\mathrm{H}^1_{\mathrm{dR}}(X) \otimes_{\mathbf{R}} \mathbf{C}$, and so the map in question is injective. Since both spaces have complex dimension $2g$, it is also surjective. □

Proposition 3. *Let $p \colon \mathrm{H}^1_{\mathrm{dR}}(X) \otimes_{\mathbf{R}} \mathbf{C} \to V$ be the projection map. Then p induces an isomorphism of real vector spaces $\mathrm{H}^1_{\mathrm{dR}}(X) \to V$.*

Proof. Suppose α is an element of $\mathrm{H}^1_{\mathrm{dR}}(X)$. Then in the decomposition $\alpha = \omega + \overline{\eta}$ with $\omega, \eta \in V$, we must have $\omega = \eta$, since $\alpha = \overline{\alpha}$. It follows that $\omega \mapsto \omega + \overline{\omega}$ is the inverse to p. □

Proposition 4. *Let $\alpha, \beta \in \mathrm{H}^1_{\mathrm{dR}}(X)$. Let $\omega = p(\alpha)$ and $\eta = p(\beta)$. Then*

$$\int_X \alpha \wedge \beta = 2\,\mathrm{Re} \int_X \omega \wedge \overline{\eta}$$

Proof. We have $\alpha = \omega + \overline{\omega}$ and $\beta = \eta + \overline{\eta}$, and so $\alpha \wedge \beta = \omega \wedge \overline{\eta} + \overline{\omega} \wedge \eta = 2\,\mathrm{Re}(\omega \wedge \overline{\eta})$. □

We define a Hermitian form H on V by

$$H(\omega, \eta) = 2i \int \omega \wedge \overline{\eta}.$$

The factor of i is required for the identity $H(\omega, \eta) = \overline{H(\eta, \omega)}$. The above proposition says that for $\alpha, \beta \in \mathrm{H}^1_{\mathrm{dR}}(X)$, we have $\int_X \alpha \wedge \beta = \mathrm{Im}(H(p(\alpha), p(\beta))$.

These are notes for Math 679, taught in the Fall 2013 semester at the University of Michigan by Andrew Snowden.

1.2 Definition of the Jacobian

Let $L = \mathrm{H}_1(X, \mathbf{Z})$. Given $\gamma \in L$ and $\omega \in V$, we can integrate ω over γ and get a number. For fixed γ, this defines a linear map $V \to \mathbf{C}$, and so we have a natural map $i\colon L \to V^*$.

Proposition 5. *$i(L)$ is a lattice in V^*.*

Proof. Let $i_{\mathbf{R}}$ denote the induced map $L \otimes \mathbf{R} \to V^*$. The real dual of V^* is natural isomorphic to V, where an element $v \in V$ induces a linear map $V^* \to \mathbf{R}$ by taking the real value of the usual pairing. The dual of $i_{\mathbf{R}}$ is thus a real-linear map $V \to L^\vee \otimes \mathbf{R}$. Identifying $L^\vee \otimes \mathbf{R}$ with $\mathrm{H}^1_{\mathrm{dR}}(X)$, this map takes ω to $\omega + \overline{\omega}$. Indeed, the image of ω is supposed to be a real 1-form α such that $\mathrm{Re} \int_\gamma \omega = \int_\gamma \alpha$ for any $\gamma \in L$, and it is clear that $\alpha = \omega + \overline{\omega}$ does the job. Thus $i_{\mathbf{R}}$ is an isomorphism; indeed, it is the inverse to p. □

Definition 6. The Jacobian of X, denoted $\mathrm{Jac}(X)$, is the complex torus V^*/L. □

We have a natural conjugate-linear isomorphism $j\colon V \to V^*$ given by $j(\omega) = H(-, \omega)$. Define a Hermitian form H^* on V^* by $H^*(\lambda, \mu) = H(j^{-1}(\mu), j^{-1}(\lambda))$; the order is switched so that H^* is linear in its first slot.

Proposition 7. *$\mathrm{Im}\, H^*(i(\gamma), i(\gamma')) = \langle \gamma, \gamma' \rangle$ for any $\gamma, \gamma' \in L$, where $\langle , \rangle$ is the intersection pairing on L.*

Proof. The two pairings $\mathrm{Im}\, H^*$ and $\langle , \rangle$ both induce real skew-symmetric pairings on $L \otimes \mathbf{R}^*$. These can be transferred to the dual space, which is identified with $\mathrm{H}^1_{\mathrm{dR}}(X)$. The first is then given by $(\alpha, \beta) \mapsto \mathrm{Im}\, H(p(\alpha), p(\beta))$, while the second by $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$. The equality of these two pairings has already been established. □

Corollary 8. *$\mathrm{Jac}(X)$ is canonically a principally polarized abelian variety.*

1.3 Basic properties

Some elementary properties of $\mathrm{Jac}(X)$:

- The tangent space to $\mathrm{Jac}(X)$ at the identity is canonically isomorphic to $V^* = \mathrm{H}^1(X, \mathcal{O})$ (the identification here is Serre duality).
- Dually, we see that the cotangent space to $\mathrm{Jac}(X)$ at 0 is V. It follows that $\mathrm{H}^0(\mathrm{Jac}(X), \Omega^1) = V = \mathrm{H}^0(X, \Omega^1)$.
- We have a natural isomorphism $\mathrm{H}_1(\mathrm{Jac}(X), \mathbf{Z}) = \mathrm{H}_1(X, \mathbf{Z})$.
- Fix a point $x \in X$. We then get a map $f_x\colon X \to \mathrm{Jac}(X)$ as follows. For $y \in X$, choose a path ρ from x to y in X. We then get an element of V^* by integrating over ρ. The choice of ρ is not unique, but the difference of any two choices lies in L, so the resulting elements of V^* is well-defined up to $i(L)$. The map f_x takes y to this element of $V^*/i(L)$.
- One can show that f_x^* induces an isomorphism $\mathrm{H}^0(\mathrm{Jac}(X), \Omega^1) \to \mathrm{H}^0(X, \Omega^1)$.

We now give perhaps the most important property of $\mathrm{Jac}(X)$. Let $\mathrm{Pic}(X)$ denote the group of isomorphic classes of line bundles on X, and let $\mathrm{Pic}^0(X)$ be the subgroup consisting of those of degree 0.

Proposition 9. *We have a natural isomorphism $\mathrm{Jac}(X) \to \mathrm{Pic}^0(X)$.*

Proof. Consider the exponential sequence on X:

$$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0.$$

Taking cohomology, we obtain an exact sequence

$$0 \to \mathrm{H}^1(X, \mathcal{O})/\mathrm{H}^1(X, \mathbf{Z}) \to \mathrm{H}^1(X, \mathcal{O}^\times) \to \mathrm{H}^2(X, \mathbf{Z}) \to 0.$$

Identifying $\mathrm{H}^1(X, \mathcal{O})$ with V^* by Serre duality, the group on the left is $\mathrm{Jac}(X)$. The group in the middle is $\mathrm{Pic}(X)$. We have a natural identification $\mathrm{H}^2(X, \mathbf{Z}) = \mathbf{Z}$, under which the right map is the degree map $\mathrm{Pic}(X) \to \mathbf{Z}$. Thus $\mathrm{Jac}(X)$ maps isomorphically to $\mathrm{Pic}^0(X)$. □

2 Algebraic theory

A good reference for this section is Chapter III of Milne's notes on abelian varieties.

2.1 Attempt at a definition

The definition of the Jacobian in the complex case was inherently analytic, and does not carry over to the algebraic case. However, we can use one of the results we proved about the Jacobian, namely that its points parametrize degree 0 line bundles on X, to give a definition valid over any field. This is similar to our previous discussion of the dual abelian variety.

From now on, we fix a field k, and let X/k be a curve. We want to give $\mathrm{Pic}^0(X)$ an algebraic structure. To do this, we must make sense of families of degree 0 line bundles on X over a base scheme T. This is not hard to do: such a family is just a line bundle on $X_T = X \times T$, which restricts to degree 0 in each fiber. Let $F(T)$ be the set of isomorphism classes of such bundles. One might then hope that F is representable, and define the Jacobian in this manner.

2.2 First obstruction to representability

Unfortunately, F is not representable. There are two obstacles to representability. The first is that line bundles on T cause problems. To see this, suppose F were represented by some variety J. Let L be a line bundle on T, and write $p\colon X_T \to T$ for the projection. The line bundle $p^*(L)$ on X_T is trivial on each fiber, and therefore of degree 0 in fiber, and thus belongs to $F(T)$. There should therefore be a map $f\colon T \to J$ such that $p^*(L)$ is isomorphic to $f^*(\mathcal{L})$, where $\mathcal{L}$ is some universal bundle on J. However, since $p^*(L)$ is trivial in each fiber, f must map all of T to the same point (the one corresponding to the trivial bundle on X), and so $f^*(\mathcal{L})$ is trivial. But $p^*(L)$ need not be trivial.

This problem can be fixed by simply killing all the bundles that come from T. Precisely, define $G(T)$ to be the quotient of $F(T)$ by the subgroup $p^*(\mathrm{Pic}(T))$. Then the above paragraph shows that we should work with G instead of F.

2.3 Second obstruction to representability

However, F suffers from another problem which prevents it, and G, from being representable: it is not necessarily a sheaf. In fact, if k'/k is a Galois extension with group Γ then the natural map $F(k) \to F(k')^\Gamma$ need not be a bijection, which is a requirement for representability. (Note that F and G have the same field-value points, so this fails for G as well.) Precisely, we have the following picture:

Proposition 10. *Let k'/k be a Galois extension of group Γ. Then there is a natural exact sequence*

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_{k'})^{\Gamma} \to \mathrm{Br}(k)$$

In particular, given $L \in \mathrm{Pic}(X_{k'})$ there is an obstruction in $\mathrm{Br}(k)$ measuring the failure of L to descend to X.

Proof. We first show that the left map is injective. In other words, if L and L' are two line bundles on X which are isomorphic over $X_{k'}$, then L and L' are isomorphic. Let i be an isomorphism over $X_{k'}$. For $\sigma \in \Gamma$, the map i^{σ} is also an isomorphism $L \to L'$, and thus differs from i by an element c_{σ} of $\mathrm{Aut}(L) = (k')^{\times}$. One easily sees that c satisfies the cocycle condition, and thus defines an element of $\mathrm{H}^1(\Gamma, (k')^{\times})$, which vanishes by Hilbert's Theorem 90. Thus c is a coboundary, i.e., of the form $c_{\sigma} = (\sigma\alpha)/\alpha$ for some $\alpha \in (k')^{\times}$. One easily sees that $\alpha^{-1}i$ is a Galois-invariant isomorphism $L \to L'$ over $X_{k'}$, and thus descends to X.

We now construct an element of $\mathrm{Br}(k)$ measuring the obstruction of an element of $\mathrm{Pic}(X_{k'})^{\Gamma}$ to come form $\mathrm{Pic}(X)$. The basic reason such an obstruction exists is because an element of $\mathrm{Pic}(X_{k'})^{\Gamma}$ is a line bundle L on $X_{k'}$ such that $\sigma^*(L)$ is isomorphic to L for each $\sigma \in \Gamma$, but these isomorphisms are not required to satisfy any sort of compatibilities, which is needed for descent. In fact, the failure of the compatibilities defines a 2-cocycle which gives the Brauer obstruction. Suppose $L \in \mathrm{Pic}(X_{k'})^{\Gamma}$, and for each $\sigma \in \Gamma$ choose an isomorphism $i_{\sigma} \colon L \to \sigma^*(L)$. Then $\sigma^*(i_{\tau}) \circ i_{\sigma}$ and $i_{\sigma\tau}$ are two isomorphisms $L \to (\sigma\tau)^*L$, and thus differ by an element $c_{\sigma,\tau}$ of $\mathrm{Aut}(L) = (k')^{\times}$. It is easy to see that c satisfies the 2-cocycle condition, and thus defines an element of $\mathrm{H}^2(\mathrm{Gal}(k'/k), (k')^{\times}) \subset \mathrm{Br}(k)$. If this 2-cocycle is a coboundary, then the choice of i's can be modified to give descent data on L, and L belongs to $\mathrm{Pic}(X)$. This completes the proof. ☐

Example 11. Take X to be a curve which is isomorphic to $\mathbf{P}^1$ over k' but not over k, e.g., the curve over $k = \mathbf{R}$ given by $X^2 + Y^2 = -Z^2$. Then $\mathrm{Pic}(X_{k'})$ is isomorphic to $\mathbf{Z}$, and thus contains $\mathrm{Pic}(X)$ with finite index, and so Γ acts trivially on $\mathrm{Pic}(X_{k'})$. But the bundle $\mathcal{O}(1)$ on $X_{k'}$ does not descend to X, as this would give an isomorphism $X \to \mathbf{P}^1$ over k. ☐

Remark 12. Suppose k is a finite extension of $\mathbf{Q}_p$ and k' is an algebraic closure of k. Then $\mathrm{Br}(k) = \mathbf{Q}/\mathbf{Z}$, and Lichtenbaum showed that the image of the the map $\mathrm{Pic}(X_{k'})^{\Gamma} \to \mathrm{Br}(k)$ is $N^{-1}\mathbf{Z}/\mathbf{Z}$, where N is the gcd of the degrees of divisors on X. Thus $\mathrm{Pic}(k) = \mathrm{Pic}(k')^{\Gamma}$ if and only if X has a divisor of degree 1 defined over k. ☐

Remark 13. We have not actually give an example where a line bundle of degree 0 fails to descend, which is the case of interest (as $F(k') = \mathrm{Pic}^0(X_{k'})$). I believe such an example exists if X is a genus 1 curve over a finite extension of $\mathbf{Q}_p$ without a point. ☐

2.4 The case where a rational point exists

In fact, the failure of G to satisfy descent only occurs when X has not k-rational points. To see this, suppose X has a k-rational point x. Define $\mathcal{G}_x(T)$ to be the category of pairs (L, i) where L is a fiberwise degree 0 line bundle on X_T, and i is an isomorphism of $L|_{\{x\} \times T}$ with the trivial bundle $\mathcal{O}_T$. Define $G_x(T)$ to be the set of isomorphism classes in $\mathcal{G}_x(T)$. The key point is that objects of $\mathcal{G}_x(T)$ are rigid: they have no automorphisms. This means that if an isomorphism class is invariant, then it has canonical descent data. It follows that G_x is a sheaf. On the other hand, we have the following lemma:

Lemma 14. *The natural map $G_x \to G$, given by forgetting i, is an isomorphism.*

Proof. If L and L' belong to G_x and $L \cong L' \otimes p^*(L'')$ for some line bundle on L'' then, restricting to $\{x\} \times T$, one sees that L'' is trivial, and so $L \cong L'$; this proves injectivity. As for surjectivity, suppose L is a line bundle on X_T and let L_0 be its restriction to $\{x\} \times T$. Then $L \otimes p^*(L_0^{-1})$ is naturally an element of G_x mapping to L in $G(T)$. □

We thus see that, when X has a k-point, G is a sheaf.

Theorem 15. *Suppose X has a k-point. Then the sheaf G is representable. The representing scheme is denoted* $\mathrm{Jac}(X)$*, and called the Jacobian of X.*

If X does not have a point then G is not necessarily a sheaf, and thus not necessarily representable. However, one can replace G with its sheafification, and this turns out to be representable. Thus one can define the Jacobian of X even when X does not have a point.

2.5 Construction of the Jacobian

We now sketch the proof of the representability of G in the case that X has a k-rational point x. Let $X^{(r)}$ be the rth symmetric power of X, i.e., the quotient of X^r by the action of the symmetric group S_r. Points on $X^{(r)}$ defined over k' can be identified with effective divisors on $X_{k'}$ of degree r. We will consider $X^{(g)}$, where g is the genus of X.

Let D and D' be effective divisors of degree g on X. The Riemann–Roch theorem then implies that $\ell(D + D' - g[x]) \geq 1$. By semi-continuity, the locus $U \subset X^{(g)} \times X^{(g)}$ where equality holds is open, and it is not difficult to show that it is non-empty. (Taking $D' = g[x]$, one must find an effective divisor D of degree g with $\ell(D) = 1$, or, equivalently $\ell(K - D) = 0$. Simply pick g points $x_1, \ldots, x_g$ of X such that the restriction map $\mathrm{H}^0(X, \Omega^1) \to \prod_{i=1}^{g} T^*_{x_i}$ is an isomorphism.) Given $(D, D') \in U$, there is thus a non-zero meromorphic function f on X, unique up to scaling, such that $D'' = \mathrm{div}(f) + D + D' - g[x]$ is effective. We define a map $U \to X^{(g)}$ by taking (D, D') to D''. By working systematically with families of divisors, one shows that this is a map of schemes.

One can regard the above map as a rational map $X^{(g)} \times X^{(g)} \dashrightarrow X^{(g)}$. As such, it satisfies the axioms to be a group (it is a group object in the category of varieties with rational maps). Weil showed that any such rational group variety can be upgraded to an actual group variety. Precisely, there exists a group variety J (unique up to isomorphism) and a unique isomorphism of rational group varieties $X^{(g)} \dashrightarrow J$

Finally, one must show that J represents G. One first shows that J is proper, and so the rational map $X^{(g)} \dashrightarrow J$ is an actual map. Then, one defines a map $f\colon \mathrm{Div}^0(X) \to J$ as follows. If D is a degree 0 divisor such that $D + g[x]$ is effective, then one regards $D + g[x]$ as an element of $X^{(g)}$ and takes its image in J. If $D + g[x]$ is not effective, then one finds a degree 0 divisor D' such that $D + D' + g[x]$ and $D' + g[x]$ are both effective, and defines $f(D) = f(D + D') - f(D')$. Working with families of divisors, f gives a map of functors $G \to J$. One then verifies that it is a bijective on T-points.

2.6 Basic properties

Many of the basic properties satisfied in the analytic case remain true in the algebraic case.

- One can show that $T_0(\mathrm{Jac}(X)) = \mathrm{H}^1(X, \mathcal{O})$ using the functor of points of $\mathrm{Jac}(X)$ and the interpretation of the tangent space in terms of dual numbers.

- From this, one finds that $\mathrm{H}^0(\mathrm{Jac}(X), \Omega^1)$ is naturally isomorphic to $\mathrm{H}^0(X, \Omega^1)$.

- One again has a map $f_x\colon X \to \mathrm{Jac}(X)$ given a base point $x \in X(k)$. On field points, this takes a point $y \in X(k)$ to the degree 0 divisor $[y] - [x]$. On T-points, it does the same thing, but one must use a relative notion of divisor.

- By definition, $\mathrm{Jac}(X)(k)$ is isomorphic to $\mathrm{Pic}^0(X)$.

One again has a comparison between the first (co)homology groups of X and $\mathrm{Jac}(X)$, though this now involves cohomology. This is most easily seen using Kummer theory. Suppose n is prime to the characteristic of k, so that we have an exact sequence of sheaves on the étale site of X:

$$0 \to \mu_n \to \mathbf{G}_m \to \mathbf{G}_m \to 0$$

Taking cohomology over $\overline{k}$, and using the fact that every element of $\overline{k}^\times$ is an nth power, we see that $\mathrm{H}^1(X_{\overline{k}}, \mathbf{G}_m)[n] = \mathrm{H}^1(X_{\overline{k}}, \mu_n)$. Now, $\mathrm{H}^1(X_{\overline{k}}, \mathbf{G}_m) = \mathrm{Pic}(X_{\overline{k}})$; since all torsion in this group is of degree 0, we see that $\mathrm{H}^1(X_{\overline{k}}, \mathbf{G}_m)[n] = \mathrm{Jac}(X)[n](\overline{k})$. Replacing n with ℓ^n and taking an inverse limit, we find $T_\ell(\mathrm{Jac}(X)) = \mathrm{H}^1(X_{\overline{k}}, \mathbf{Z}_\ell(1))$, where the (1) is a Tate twist.

2.7 In relative situations

Suppose $C \to S$ is a family of smooth projective curves with geometrically connected fibers. One can then define a functor G just as we did above. When C has a section over S, this functor is representable by an abelian scheme $\mathrm{Jac}(C)$, which one would call the relative Jacobian.

One reason this is relevant for us is as follows. Suppose R is a DVR with fraction field K, let X/K be a curve, and let J be its Jacobian. Suppose we can find a nice model of X over R (smooth, projective, geometrically connected fibers). Then the relative Jacboian of this model is an abelian scheme extending J. This shows that J has good reduction.