Hidden Action Contract: Example

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SI 680

- **Initializations**

Problem statement

A principal seeks an agent to sell her new software product. Sales revenues, x are observable. Suppose the agent can exert one of two effort levels, $e \in \{0, 1\}$. If the agent makes effort $e = 1$ he bears cost C; there is no cost for effort $e = 0$. The agent experiences utility $U(t, x, e) = u(t) - \text{eC}$ where t is the amount of transfer (compensation) from the principal to the agent. The function $u(\cdot)$ is increasing and concave ($u' > 0$, $u'' < 0$). The agent has a reservation utility (utility value of his next best opportunity) of U_0.

Suppose sales can take on one of two values, $x \in \{x_L, x_H\}$. The level of sales depends on the agent's effort, and also on an unobserved random variable. The discrete probability distribution for sales is $F_e(x)$; for example, the probability of observing high output when effort is low is labeled $F_0(x_L)$. The principal is risk-neutral and experiences utility $\Pi(t, x) = 200 + x - t$.

The principal offers the agent a take-it-or-leave-it contract. If effort is observable by the principal and verifiable by a third party, we say the problem has *symmetric information*. If effort is unobservable by the principal or unverifiable by a third party, we say the problem has *asymmetric information*, and in particular, is a problem of hidden information.

General formulation: symmetric information

Effort is verifiable, so transfers can depend on effort, $t = t(e)$. Since the principal is risk neutral and the agent is risk averse, the principal will not make the transfer contingent on x, since that needlessly makes the agent bear some of the uncontrollable risk. Therefore, the transfer will take on one of two values: $t \in \{t_0, t_1\}$ for $e = 0$ and $e = 1$, respectively.
If the principal wants the agent to exert effort $e = 1$, she offers a take-it-or-leave-it contract requiring $e = 1$ and paying $t = t_1$. The agent will not sign the contract unless his utility is at least as high as his reservation utility.

$$\text{(PC)} \quad u(t_1) - C \geq U_0 \qquad (1)$$

Then the principal chooses t_1 to solve

$$\text{(MAX)} \quad \max_{t_1} 200 + E_1[x] - t_1 \quad \text{s.t. (PC)} \qquad (2)$$

The principal can set up and solve the similar problem to find the best contract for an agent who exerts $e = 0$. The optimal contract overall is the one of these two that yields the higher expected profits.

General formulation: asymmetric information

Effort is not verifiable, so transfers can only depend on sales, *t* = *t(x)*. Thus, transfers may be random, so the agent is concerned about expected utility. The agent will not sign a contract unless his expected utility is at least as high as his reservation utility (participation constraint).

$$(PC') \quad E_e[u(t(x))] - C \geq U_0 \tag{3}$$

where E_e is the expectation taken with respect to the distribution of sales when effort takes on value *e*.
Further, since the principal cannot verify effort, but wants the agent to exert effort *e* = 1, it must be that the agent gets higher expected utility from *e* = 1 than from *e* = 0 (incentive constraint).

$$(IC') \quad E_1[u(t(x))] - C \geq E_0[u(t(x))] \tag{4}$$

Then the principal chooses *t(x)* to solve

$$(MAX') \quad \max_{t(x)} 200 + E_1[x - t(x)] \quad \text{s.t. } (PC') \text{ and } (IC') \tag{5}$$

Numerical example: symmetric information

We need to make functional and numerical assumptions to solve an example. Suppose that the probability disribution of sales, given agent effort, is:

$$F_e(x_L) = k - e\,q \tag{6}$$

for $q \in (0, 1]$ and $k \in [q, 1]$. Since there are only two sales levels, $F_e(x_H) = 1 - F_e(x_L)$.

Further, suppose that the cost of effort is $C = 2$, reservation utility is $U_0 = 3$, $k = 0.75$, $q = 0.5$, $x_L = 50$, $x_H = 200$, and $u(\cdot) = \ln(\cdot)$.
Now, the principal's problem if she wants the agent to exert effort is:

$$\begin{aligned} &(MAX) \quad \max_{t_1} 200 + (1 + q - k)\,200 + (k - q)\,50 - t_1 = \\ &200 + .75 * 200 + .25 * 50 - t_1 \\ &\text{s.t. } (PC) \quad \ln(t_1) - 2 \geq 3 \end{aligned} \tag{7}$$

We could solve this by forming the Lagrangian and working through the first-order conditions, but there is a simpler way that involves relying on our understanding of the problem: thus thinking through this alternative method will improve our understanding. The key: does the (PC) bind (solve as an equality)? Suppose it didn't: then $\ln(t_1) \geq 5$. But then there is a $t_1' = t_1 - \epsilon$ that would increase the principal's utility (since utility is decreasing in t_1) while still satisfying the (PC). Lowering t_1 until (PC) binds satisfies the constraint and maximizes principal utility. Therefore, $t_{1S} = \text{Exp}[5]$ is the best contract (for an agent who is required to make effort *e* = 1) (I use subscript *S* to indicate optimal values for the symmetric information problem).

```
t1S = Exp[5] // N
```

```
148.413
```

That is, if the principal wants effort, she offers a take-it-or-leave-it contract with {*e*, *t*} = {1, 148.4}.

To determine the overall optimal symmetric information contract, we need to also solve for the best contract when the agent does not make effort (*e* = 0), and then choose the contract under which the principal's profits are highest. (This could have been solved by simultaneously optimizing over both *t* and *e*, but when there are only two possible levels of *e* it may be a bit simpler to solve for the two different types of contracts separately and then compare profits.) I leave this calculation as an exercise.

Numerical example: asymmetric information

Now we can solve for the best asymmetric information contract if the principal wants the agent to make effort. Substituting the parameter values above into expressions (3), (4) and (5), the principal's problem if she wants the agent to exert effort is:

$$\text{(MAX')} \quad \max_{\{t_L, t_H\}} 200 + .75 * (200 - t_H) + .25 * (50 - t_L)$$

$$\text{s.t. (PC')} \quad .75 \ln(t_H) + .25 \ln(t_L) - 2 \geq 3 \tag{8}$$

$$\text{(IC')} \quad .75 \ln(t_H) + .25 \ln(t_L) - 2 \geq .25 \ln(t_H) + .75 \ln(t_L)$$

Let's set up the Lagrangean for this problem to solve it:

```
lagr[th_, tl_] = 200 + .75 (200 - th) + .25 (50 - tl) +
   lam1 (.75 Log[th] + .25 Log[tl] - 5) + lam2 (.5 Log[th] - .5 Log[tl] - 2);
```

The first order conditions with respect to t_H and t_L are:

```
phiH = D[lagr[th, tl], th]
phiL = D[lagr[th, tl], tl]
```

$$-0.75 + \frac{0.75\,\text{lam1}}{\text{th}} + \frac{0.5\,\text{lam2}}{\text{th}}$$

$$-0.25 + \frac{0.25\,\text{lam1}}{\text{tl}} - \frac{0.5\,\text{lam2}}{\text{tl}}$$

Now, let's figure out which constraints are binding. This is a helpful first step because a binding constraint is an equality, and it's easier to solve a system of equalities than a system of inequalities.

First, is (PC') binding? Suppose not: then by definition its Lagrange multiplier (λ_1) will be zero. Setting the two first order conditions equal to zero when $\lambda_1 = 0$ leads to the immediate conclusion that $t_L = t_H = 1/\lambda_1$, which means a constant payment, or a pure insurance contract for the agent (no risk). But what happens when we put this result into the (IC') constraint? Let t be the constant value of the transfer; we have

$$\text{(IC')} \quad \ln(t) - 2 \geq \ln(t) \tag{9}$$

which is impossible, so the premise must be false: the (PC') constraint must bind.

What about (IC')? Consider the same type of argument we made above for the symmetric case: if (IC') were not binding, we could subtract some amount from *both* t_H and t_L, but since the principal's utility is decreasing in both transfer amounts, that would increase the maximand. Therefore, we should subtract a larger and larger amount until the (IC') is just binding.

Conclusion so far: both (PC') and (IC') are binding. That means we have two equalities in two unknowns, and we can simply solve them for the optimal values of t_H and t_L. You can do this the usual plug-and-chug way (e.g., solve (PC') for t_H as a function of t_L, plug the result into (IC') to solve for t_L, then substitute back into (PC') to solve for t_H). I'll take the easy way and let *Mathematica* find the solutions for me. First, a trick that is good whichever way you solve the equations: it's messy to solve equations involving logarithms. But, since t_H and t_L always and only appear as logarithms, simply do a change of variables; replace with $u_H = \text{Log}[t_H]$ and $u_L = \text{Log}[t_L]$. We'll get the solution in terms of utility levels for the agent, but we can always reverse the process (e.g., $t_H = \text{Exp}[u_H]$) to get back the values of the transfers.

```
pc[uh_, ul_] := .75 uh + .25 ul - 5;
ic[uh_, ul_] := 0.5 uh - 0.5 ul - 2;
contractAsymm = Solve[{pc[uh, ul] == 0, ic[uh, ul] == 0}, {uh, ul}]
```

```
{{uh → 6., ul → 2.}}
```

```
th = First[Exp[uh] /. contractAsymm]
tl = First[Exp[ul] /. contractAsymm]
```

```
403.429

7.38906
```

Notice that, as we expect, one of the transfers is higher, and one is lower, than in the single-payment (full-insurance) symmetric information case $\left(t_1{}^S = 148.4\right)$.

Is principal utility higher or lower with asymmetric information? What do you expect?

```
principalUSymm = 200 + .75 * 200 + .25 * 50 - t1S
principalUAsymm = 200 + .75 * (200 - th) + .25 * (50 - tl)
```

```
214.087

58.0811
```

So, utility is lower when there is asymmetric information: hopefully this is what you predicted. Because of the conflict of interest, there is necessarily a cost to the principal: the information rent she must pay. Further, there is an inefficiency: the agent who is risk-averse must now bear some risk (different payments in different states of the world), and since he is just getting his reservation utility (that's what (PC') binding means), the principal must be losing the value of this inefficiency.

As with the symmetric case, we do not know if this is the *overall optimal* contract: that depends on how much utility the principal gets from the best contract she can write if she only wants the agent to exert effort $e = 0$. Again, I leave solving for this contract to you as an exercise.

Graphical solution of the asymmetric problem

Recall the asymmetric information problem from above (expression (8), repeated here):

To solve this graphically, we need to graph the maximand, graph the constraints, and find the values of t_L and t_H that maximize the maximand in the space permitted by the constraints.

I'm going to use two tricks. First, as above, we'll find it easier to work with the levels of agent utility in the high and low states (u_L and u_H), rather than with the levels of transfer, so I'll use the same change of variables. Here is the restated problem (compare to expression (8):

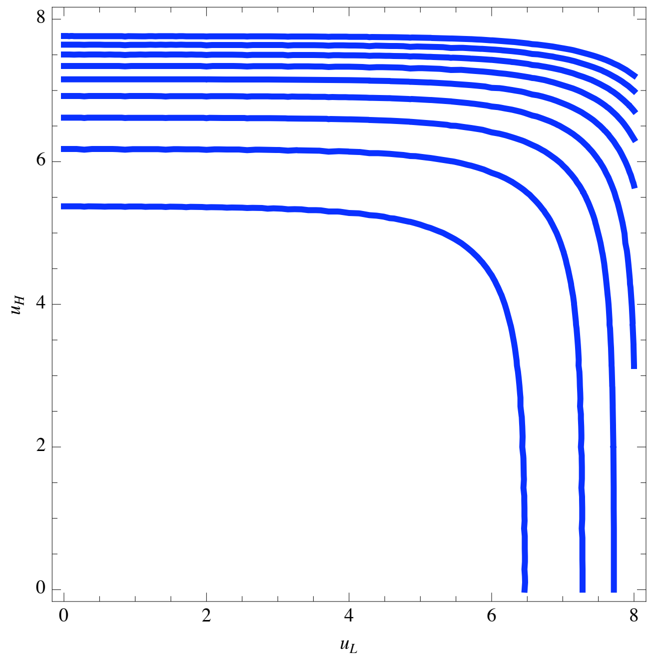
$$\text{(MAX')}\quad \max_{\{t_L, t_H\}} 200 + .75 * (200 - \text{Exp}[u_H]) + .25 * (50 - \text{Exp}[u_L])$$

$$\text{s.t. (PC')}\quad .75\, u_H + .25\, u_L - 5 \geq 0 \tag{10}$$

$$\text{(IC')}\quad 0.5\, u_H - 0.5\, u_L - 2 \geq 0$$

Since there are two unknowns, the problem lives in three dimensions (principal utility as a function of the two transfers). It's not so easy to see the solution in a 3D graph, so I'll use the usual trick: we'll plot in 2D, and for the maximand we'll plot the contours or "level curves", or the utility indifference curves. That is, we'll plot the curves that all combinations of t_L and t_H that yield a given value of principal utility.

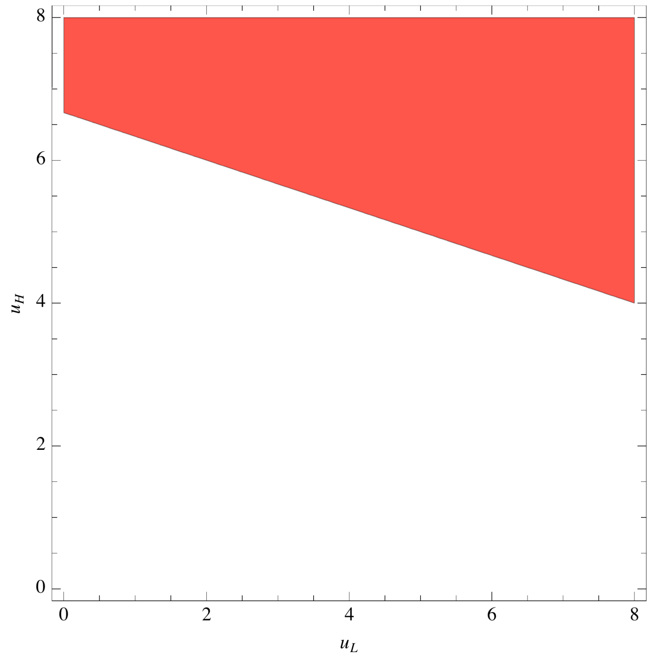
```
pUtil = ContourPlot[200 + .75 * (200 - Exp[uh]) + .25 * (50 - Exp[ul]),
  {ul, 0, 8}, {uh, 0, 8}, ContourShading → None,
  ContourStyle → Directive[Blue, Thickness[0.01]], FrameLabel → {"uL", "uH"}]
```



We know that the principal's utility is decreasing in the value of the transfers, and thus in the agent's utility of the transfers, so contours closer to the origin represent higher levels of principal utility. That is, the principal prefers to move towards the south-west of the graph.

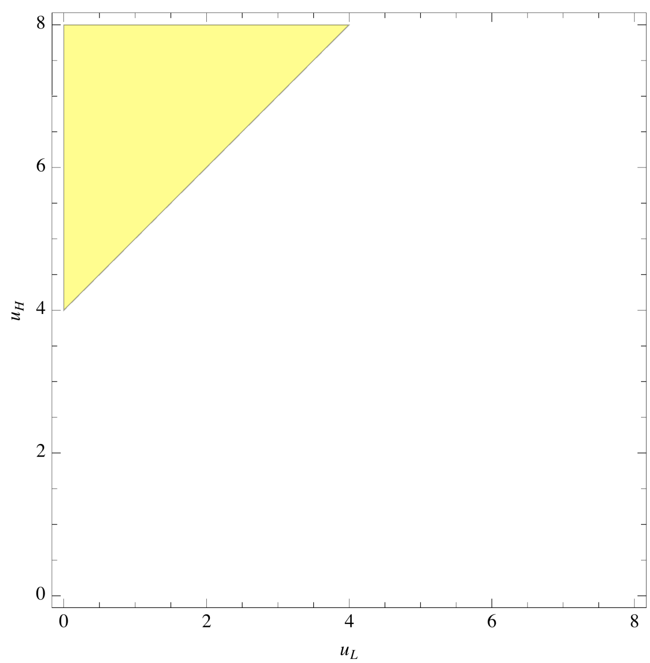
What do the constraints look like? Each is an inequality (if we are using the graphs to solve the problem, we have not necessarily figured out that the contraints are binding equalities yet). That means we want to plot the regions over which the constraints are satisfied. First (PC'):

```
pcCons = RegionPlot[.75 uh + .25 ul - 5 ≥ 0, {ul, 0, 8}, {uh, 0, 8},
  PlotStyle → Directive[Red, Opacity → .75], FrameLabel → {"uL", "uH"}]
```



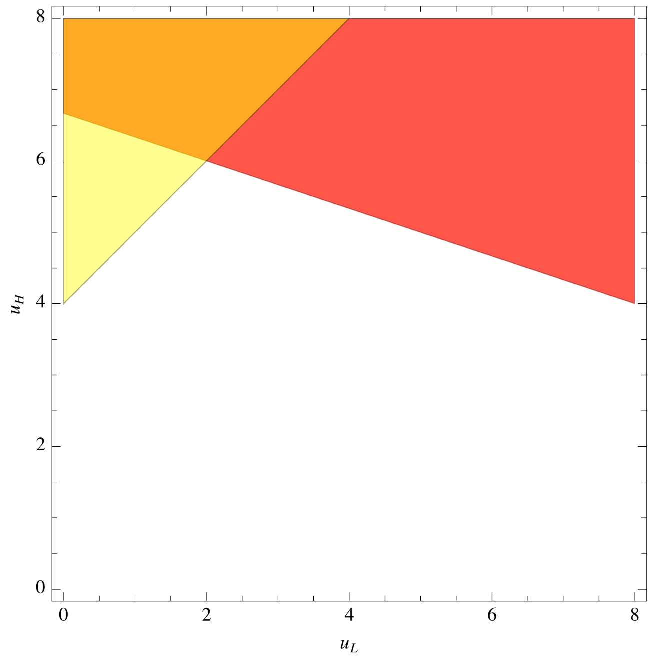
And (IC'):

```
icCons = RegionPlot[0.5 uh - 0.5 ul - 2 ≥ 0, {ul, 0, 8}, {uh, 0, 8},
  PlotStyle → Directive[Yellow, Opacity → .5], FrameLabel → {"uL", "uH"}]
```



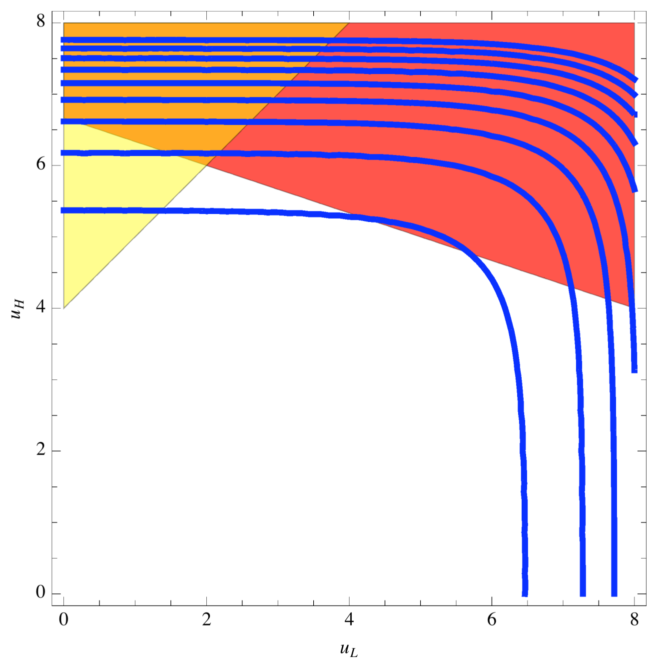
The two constraints together :

```
Show[pcCons, icCons]
```



Only values of $\{u_L, u_H\}$ that lie in *both* regions satisfy both (PC') and (IC'), so the solution to the optimal ($e = 1$) contract must be in the orange region of overlap. What happens when we graph the constraints together with the principal's indifference curves?

```
Show[pcCons, icCons, pUtil]
```



Remember that the principal's utility is increasing to the southwest: it wants to get on the blue indifference curve that is closest to the origin. That means the optimal contract will be one that has an indifference curve going right through the intersection of (PC') and (IC'). Could sketch it by hand. To get an exact plot, I'm going to use the values the principal's utility we found above in the numerical solution, then plot the principal's indifference curve for that level of utility.

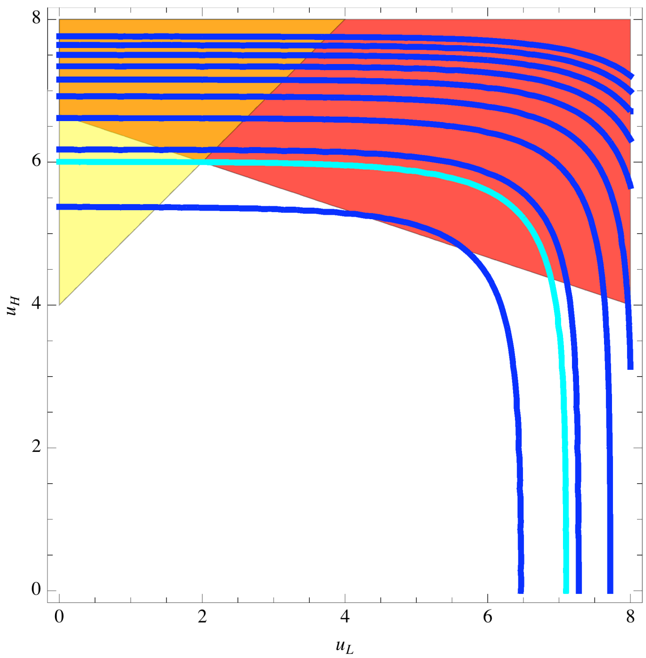
```
principalUAsymm
```

```
58.0811
```

I'll let *Mathematica* calculate the indifgerence curve when the principal's utility is equal to this level, then add it to the plot.

```
pUtilSol = ContourPlot[200 + .75 * (200 - Exp[uh]) + .25 * (50 - Exp[ul]) == principalUAsymm,
   {ul, 0, 8}, {uh, 0, 8}, ContourShading → None,
   ContourStyle → Directive[Cyan, Thickness[0.01]], FrameLabel → {"uL", "uH"}];
```

```
Show[pcCons, icCons, pUtil, pUtilSol]
```



We're done. The cyan indifference curve shows the highest utility the principal can obtain subject to the (PC') and (IC') constraints (that is, considering only points in the orange region). The optimal contract has (as we learned above), $\{u_L, u_H\} = \{2, 6\}$, which if we exponentiate to convert to the transfers when sales are low or high gives us $\{t_L, t_H\} = \{7.4, 403.4\}$. We can interpret this as a salary + bonus contract: The salesperson gets 7.4 no matter how how his sales are, and gets a bonus of 396 (for a total of 403.4) when his sales are high. The principal needs to share 396 of her additional profit with the agent if she wants him to be willing to make high (unobservable) effort.