Repeated Games and Reputation

Professor Yan Chen
Fall 2008

Some material in this lecture drawn from http://gametheory.net/lectures/level.pl
Agenda

– Finitely repeated games

– Infinitely repeated games

– Folk Theorems
  » Minmax
  » Nash-threat

– Fun project: ad auction (Next Class)
Repeated Games and Reputation

(Watson Chapter 22)
Repeated Interaction

• Empirical observations
  – People often interact in ongoing relationships
  – Your behavior today might influence actions of others in the future

• New dimension: time

• Questions
  – What if interaction is repeated?
  – What strategies can lead players to cooperate?
Definitions

• Repeated game: played over discrete periods of time (period 1, period 2, and so on)
  – \( t \): any given period
  – \( T \): total number of periods

• In each period, players play a static stage game

• History of play: sequence of action profiles
A Two-Period Repeated Game

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4,3</td>
<td>0,0</td>
<td>1,4</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>2,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Stage game, repeated once (T = 2)

Stage game NE: (A, Z), (B, Y)
### Subgame Following (A, Z)

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5,7</td>
<td>1,4</td>
<td>2,8</td>
</tr>
<tr>
<td>B</td>
<td>1,4</td>
<td>3,5</td>
<td>1,4</td>
</tr>
</tbody>
</table>

The subgame following (A,Z), with payoffs (1, 4)
Repeated Game Payoffs

All possible repeated game payoffs: larger set
Result: Any sequence of stage Nash profiles can be supported as the outcome of a SPNE. And there are more SPNE!
A Two-Period Repeated Game: Reputational Equilibrium as SPNE

• Reputational equilibrium:
  – Nonstage Nash profile in 1st period
  – Stage Nash profile in 2nd period
  – 2nd period actions contingent on outcome in first period (whether players cheat or not)

• Example:
  – Select (A, X) in 1st period
  – If player 2 chooses X in 1st period, select (A, Z) in 2nd period
  – If player 2 chooses Y or Z in 1st period, select (B, Y) in 2nd period
Infinitely Repeated Games

- Discounting ($\delta$): future payoffs not as valuable as current payoffs
  - A fixed known chance of game’s ending after each round, $p$
  - Interest rate, $r$

$$\delta = 1 - p = 1/(1+r)$$
Aside: Discounting

• Discounting:
  – Present-day value of future profits is less than value of current profits

• $r$ is the interest rate
  – Invest $1$ today → get $(1+r)$ next year
  – Want $1$ next year → invest $1/(1+r)$ today
  – Annuity paying $1$ today and $1$ every year has a net present value of $1+1/r$
Aside: Infinite Sums

\[ 1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \frac{1}{(1+r)^4} + \ldots = 1 + \frac{1}{r} \]

or:

\[ 1 + \delta + \delta^2 + \delta^3 + \ldots = \frac{1}{1-\delta} \]

• Why?

\[ s = 1 + \delta + \delta^2 + \delta^3 + \ldots \]
\[ s = 1 + \delta s \]
\[ s = \frac{1}{1-\delta} \]
### The Prisoner’s Dilemma

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>54, 54</td>
<td>72, 47</td>
</tr>
<tr>
<td>High</td>
<td>47, 72</td>
<td>60, 60</td>
</tr>
</tbody>
</table>

**Equilibrium:** $54\,K$

**Cooperation:** $60\,K$

Source: Mike Shor, gametheory.net
Prisoner’s Dilemma

• Private rationality ➔ collective irrationality

» The equilibrium that arises from using dominant strategies is worse for every player than the outcome that would arise if every player used her dominated strategy instead

• Goal:

» To sustain mutually beneficial cooperative outcome overcoming incentives to cheat

Source: Mike Shor, gametheory.net
Moving Beyond the Prisoner’s Dilemma

- Why does the dilemma occur?
  - Interaction
    » No fear of punishment
    » Short term or myopic play
  - Firms:
    » Lack of monopoly power
    » Homogeneity in products and costs
    » Overcapacity
    » Incentives for profit or market share
  - Consumers
    » Price sensitive
    » Price aware
    » Low switching costs

Source: Mike Shor, gametheory.net
Altering Interaction

• Interaction
  – No fear of punishment
    » Exploit repeated play
  – Short term or myopic play
    » Introduce repeated encounters
    » Introduce uncertainty

Source: Mike Shor, gametheory.net
Long-Term Interaction

- No last period, so no backward induction
- Use history-dependent strategies
- *Trigger strategies*:
  » Begin by cooperating
  » Cooperate as long as the rivals do
  » Upon observing a defection:
    immediately revert to a period of punishment of specified length in which everyone plays non-cooperatively
Two Trigger Strategies

• Grim trigger strategy
  – Cooperate until a rival deviates
  – Once a deviation occurs, play non-cooperatively for the rest of the game

• Tit-for-tat
  – Cooperate if your rival cooperated in the most recent period
  – Cheat if your rival cheated in the most recent period
**Trigger Strategy Extremes**

- **Tit-for-Tat is**
  - most forgiving
  - shortest memory
  - proportional
  - credible but lacks deterrence

**Tit-for-tat answers:**

*“Is cooperation easy?”*

- **Grim trigger is**
  - least forgiving
  - longest memory
  - adequate deterrence but lacks credibility

**Grim trigger answers:**

*“Is cooperation possible?”*
Why Cooperate (Against Grim Trigger Strategy)?

• Cooperate if the present value of cooperation is greater than the present value of defection

  Cooperate:
  60 today, 60 next year, 60 … 60

  Defect:
  72 today, 54 next year, 54 … 54

Firm 1

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>54, 54</td>
<td>72, 47</td>
</tr>
<tr>
<td>High</td>
<td>47, 72</td>
<td>60, 60</td>
</tr>
</tbody>
</table>

Firm 2
Payoff Stream (GTS)

profit

72

60

54

t
t+1
t+2
t+3
time

cooperate

defect

profit at t is 72, at t+1 to t+3 is 60, at t+4 is 54.
Calculus of GTS

• Cooperate if

\[
\text{PV(cooperation)} > \text{PV(defection)}
\]
\[
60\ldots60\ldots60\ldots60\ldots > 72\ldots54\ldots54\ldots54\ldots
\]
\[
60/(1-\delta) > 72 + 54 \delta/(1-\delta)
\]
\[
18\delta > 12
\]
\[
\delta > 2/3
\]

• Cooperation is sustainable using grim trigger strategies as long as \(\delta > 2/3\)
Payoff Stream (TFT)

Profit stream over time:

- Profit at time $t$: 72
- Profit at time $t+1$: 54
- Profit at time $t+2$: 60
- Profit at time $t+3$: 54

Actions:
- Cooperate: 60 profit
- Defect once: 54 profit
- Defect continuously: 47 profit
Calculus of TFT

- Cooperate if

\[
\text{PV(\text{cooperation})} > \text{PV(\text{defection})} \quad \text{and} \\
\text{PV(\text{cooperation})} > 72 \ldots 47 \ldots 60 \ldots 60 \ldots \\
60 + 60 \delta > 72 + 47 \delta \\
13 \delta > 12 \\
\delta > 12/13
\]

- Much harder to sustain than grim trigger
- Cooperation may not be likely
Trigger Strategies

• Grim Trigger and Tit-for-Tat are extremes

• Balance two goals:

  *Deterrence*

    » GTS is adequate punishment
    » Tit-for-tat might be too little

  *Credibility*

    » GTS hurts the punisher too much
    » Tit-for-tat is credible
Axelrod’s Simulation

• R. Axelrod, *The Evolution of Cooperation*
• Prisoner’s Dilemma repeated 200 times
• Game theorists submitted strategies
• Pairs of strategies competed
• Winner: Tit-for-Tat
• Reasons:
  » Forgiving, Nice, Provocable, Clear
Main Ideas from Axelrod

- Not necessarily tit-for-tat
  » Doesn’t always work

- Don’t be envious
- Don’t be the first to cheat
- Reciprocate opponent’s behavior
  » Cooperation and defection
- Don’t be too clever
Summary

• Cooperation
  » Struggle between high profits today and a lasting relationship into the future

• Deterrence
  » A clear, provicable policy of punishment

• Credibility
  » Must incorporate forgiveness

• Looking ahead:
  » How to be credible?
Another PD Example

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>4,4</td>
<td>-2,6</td>
</tr>
<tr>
<td>D</td>
<td>6,-2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

1  2
When cooperation can be sustained: grim trigger

Conditions under which cooperation can be sustained:
We check whether Grim Trigger can form a SPNE:
Suppose \( j \) plays GT. If \( i \) also plays GT, her payoff is

\[
4 + 4\delta + 4\delta^2 + 4\delta^3 + \ldots = \frac{4}{1 - \delta}.
\]

If \( i \) defects, she gets 6 in period of defection, and 0 afterwards.
Player \( i \) has an incentive to cooperate if

\[
\frac{4}{1 - \delta} \geq 6, \quad \text{or} \quad \delta \geq \frac{1}{3}
\]
• Players alternate between (C, C) and (D, C) over time, starting with (C, C)
• If either or both deviates from the alternating strategy, both will revert to the stage Nash profile, (D, D)
• Can MGT be supported as a SPNE?
Suppose 2 plays MGT. If 1 also plays MGT, 1’s payoff is
\[ PV_1 = 4 + 6\delta + 4\delta^2 + 6\delta^3 + \ldots \]
\[ = 4(1 + \delta^2 + \delta^4 + \ldots) + 6\delta(1 + \delta^2 + \delta^4 + \ldots) \]
\[ = \frac{4 + 6\delta}{1 - \delta^2}. \]

If 2 plays MGT, 2’s payoff is
\[ PV_2 = 4 - 2\delta + 4\delta^2 - 2\delta^3 + \ldots \]
\[ = 4(1 + \delta^2 + \delta^4 + \ldots) - 2\delta(1 + \delta^2 + \delta^4 + \ldots) \]
\[ = \frac{4 - 2\delta}{1 - \delta^2}. \]
(1) If 2 defects in an odd-numbered period, her payoff is 6 in this round, and 0 after: 2 has no incentive to deviate in any odd-numbered period, if

\[ \frac{4 - 2\delta}{1 - \delta^2} \geq 6, \quad or \quad 3\delta^2 - \delta - 1 \geq 0, \quad or \quad \delta \geq 0.77. \]

(2) If 2 defects in an even-numbered period, her payoff is 0 in this round, and 0 after: 2 has no incentive to deviate in any even-numbered period, if

\[ \frac{-2 + 4\delta}{1 - \delta^2} \geq 0, \quad or \quad \delta \geq 0.5. \]

Therefore, MGT can be supported as SPNE if \( \delta \geq 0.77 \).
Equilibrium Payoff Set with Discounting

- Depending on the discount factor, there are many SPNE in the repeated PD
  - (D, D) in every period
  - (GT, GT)
  - (TFT, TFT) etc.
Any payoff inside or on the edges of the diamond can be obtained as an average payoff if players choose the right sequence of actions over time.
Any point on the edges or interior of the shaded area can be supported as an equilibrium average per-period payoff, as long as the players are patient enough.
Folk Theorems

• The Nash-threat Folk Theorem:

For repeated games with stage game $G$, for any feasible payoffs $(M)$ greater than or equal to the Nash equilibrium payoffs, and for sufficiently large discount factor, there is a SPNE that has payoffs $M$. 
Applications

• Governing the Commons – The Evolution of Institutions for Collective Action by Elinor Ostrom

• International trade agreements

• eBay’s reputation system

(Check out Chapter 23)
Highlights

• Finitely repeated games
• Infinitely repeated games
• Folk theorems
• Next week:
  Games with Incomplete Information
• Fun exercise: ad words auction
Homework Assignment

- Chapter 22: #1, 2, 3, 5