Architecture 324
Structures II

Reinforced Concrete by Ultimate Strength Design

- LRFD vs. ASD
- Failure Modes
- Flexure Equations
- Analysis of Rectangular Beams
- Design of Rectangular Beams
- Analysis of Non-rectangular Beams
- Design of Non-rectangular Beams

Allowable Stress – WSD (ASD)

$$f_{actual} \leq (F.S.)F_{failure}$$

- Actual loads used to determine stress
- Allowable stress reduced by factor of safety

Ultimate Strength – (LRFD)

- Loads increased depending on type load

  γ Factors: DL=1.4 LL=1.7 WL=1.3

  U=1.4DL+1.7LL

- Strength reduced depending on type force

  ϕ Factors: flexure=0.9 shear=0.85 column=0.7

$$M_u \leq \phi M_n$$

Examples:

WSD

$$f_b \leq 0.45 f_c'$$

$$f_v \leq 0.1\sqrt{f_c'}$$

Ultimate Strength

$$M_u \leq 0.9 M_n$$

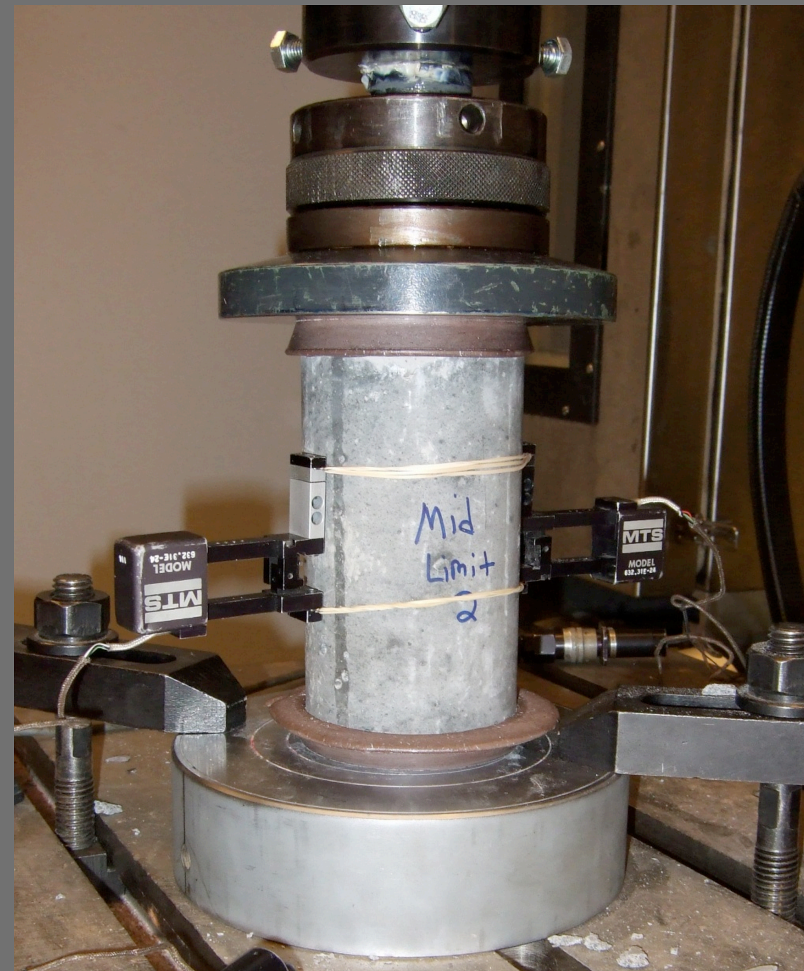
$$V_u \leq 0.85 V_n$$

$$P_u \leq 0.70 P_n$$

Strength Measurement

- Compressive strength f_c'
  - 12”x6” cylinder
  - 28 day moist cure
  - Ultimate (failure) strength

- Tensile strength f_t'
  - 12”x6” cylinder
  - 28 day moist cure
  - Ultimate (failure) strength
  - Split cylinder test
  - Ca. 10% to 20% of f’c



Photos: Source: Xb-70 (wikipedia)

Failure Modes

$$\rho = \frac{A_s}{bd}$$

- No Reinforcing
  - Brittle failure
- Reinforcing < balance
  - Steel yields before concrete fails
  - ductile failure
- Reinforcing = balance
  - Concrete fails just as steel yields
- Reinforcing > balance
  - Concrete fails before steel yields
  - Sudden failure

Source: Polyparadigm (wikipedia)

$$\rho_{\min} = \frac{200}{f_y}$$

$$\rho_{\max} = 0.75\rho_{bal}$$

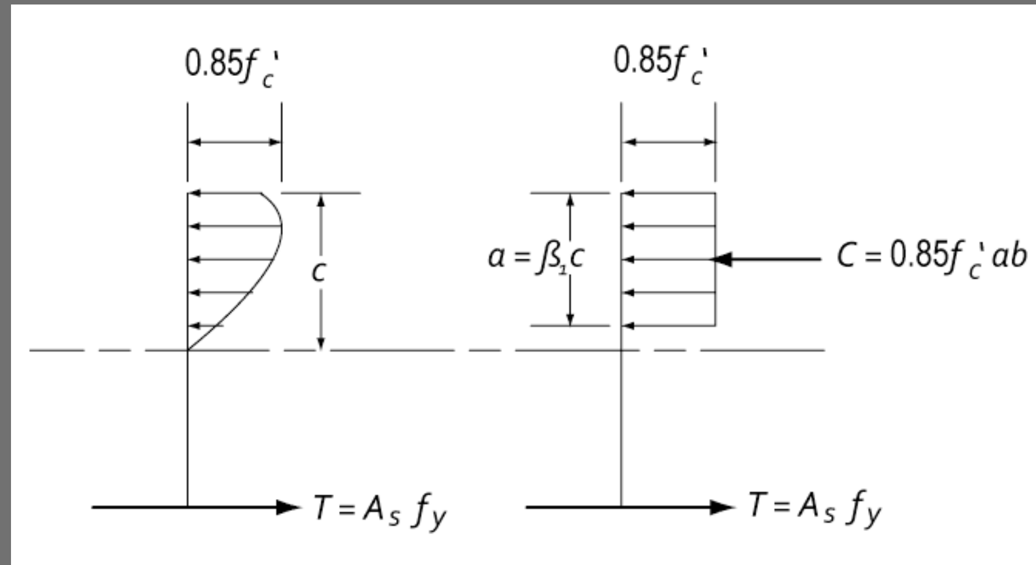
$$\rho_{bal} = \left(\frac{0.85\beta_1 f_c'}{f_y}\right)\left(\frac{87000}{87000 + f_y}\right)$$

$$\rho > \rho_{\max} \quad \text{SuddenDeath!!}$$

β_1

β_1 is a factor to account for the non-linear shape of the compression stress block.

$$a = \beta_1 c$$



f'c	β_1
0	0.85
1000	0.85
2000	0.85
3000	0.85
4000	0.85
5000	0.8
6000	0.75
7000	0.7
8000	0.65
9000	0.65
10000	0.65

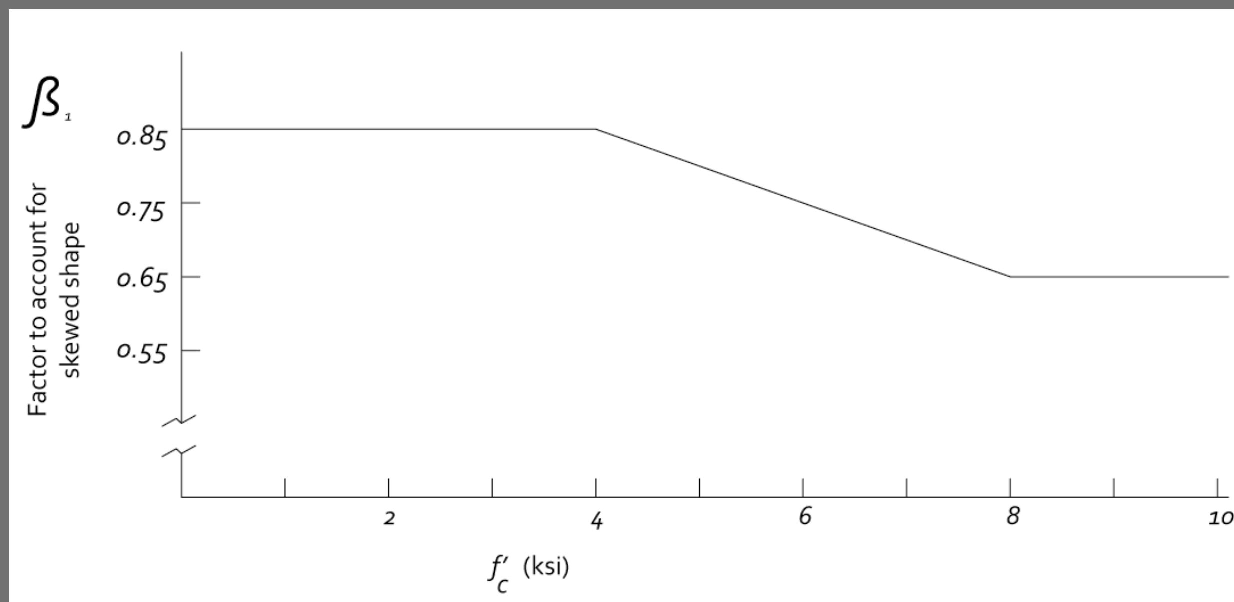


Image Sources: University of Michigan, Department of Architecture

Flexure Equations

actual stress block | ACI equivalent stress block

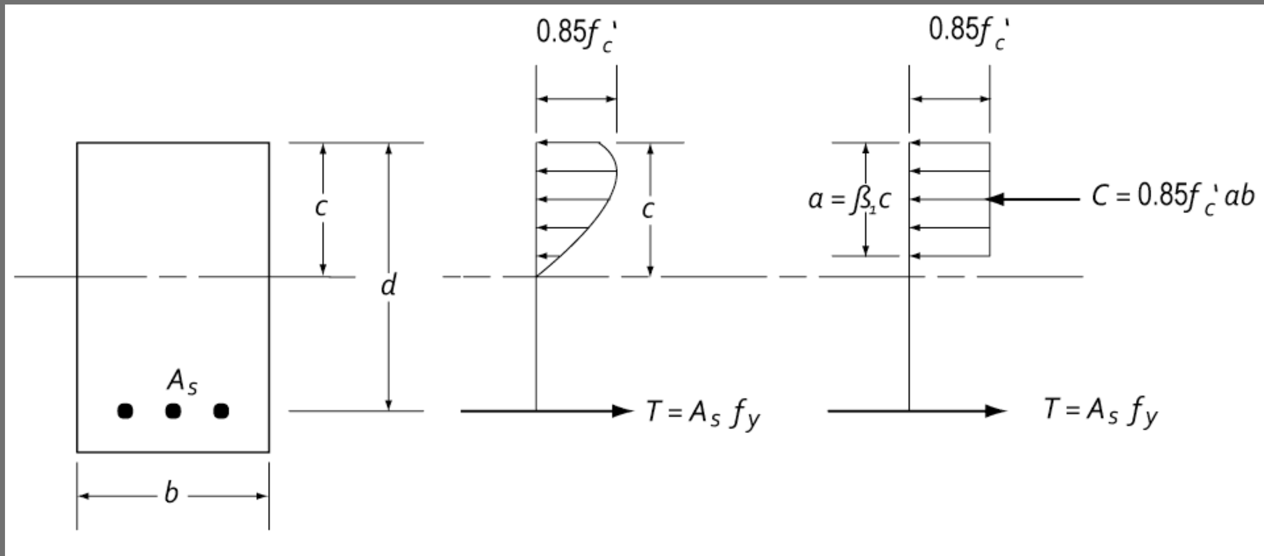


Image Sources: University of Michigan, Department of Architecture

$$C = T$$

$$0.85 f_c' ab = A_s f_y$$

solving for a,

$$a = \frac{A_s f_y}{0.85 f_c' b} = \frac{\rho f_y d}{0.85 f_c'}$$

$$\rho = \frac{A_s}{bd}$$

$$M_n = T\left(d - \frac{a}{2}\right) = A_s f_y \left(d - \frac{a}{2}\right)$$

$$M_u = \phi M_n$$

$$M_u = \phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right)$$

$$M_u = \phi A_s f_y d \left(1 - 0.59 \frac{\rho f_y}{f_c'}\right)$$

Balance Condition

From similar triangles at balance condition:

$$\frac{c}{d} = \frac{0.003}{0.003 + (f_y/E_s)} = \frac{0.003}{0.003 + (f_y/29 \times 10^6)}$$

$$c = \frac{87{,}000}{87{,}000 + f_y} d$$

Use equation for a. Substitute into $c = a/\beta_1$

$$a = \frac{\rho f_y d}{0.85 f_c'}$$

$$c = \frac{a}{\beta_1} = \frac{\rho f_y d}{0.85 \beta_1 f_c'}$$

Equate expressions for c:

$$\frac{\rho f_y d}{0.85 \beta_1 f_c'} = \frac{87{,}000}{87{,}000 + f_y} d$$

$$\rho_b = \left(\frac{0.85 \beta_1 f_c'}{f_y}\right)\left(\frac{87{,}000}{87{,}000 + f_y}\right)$$

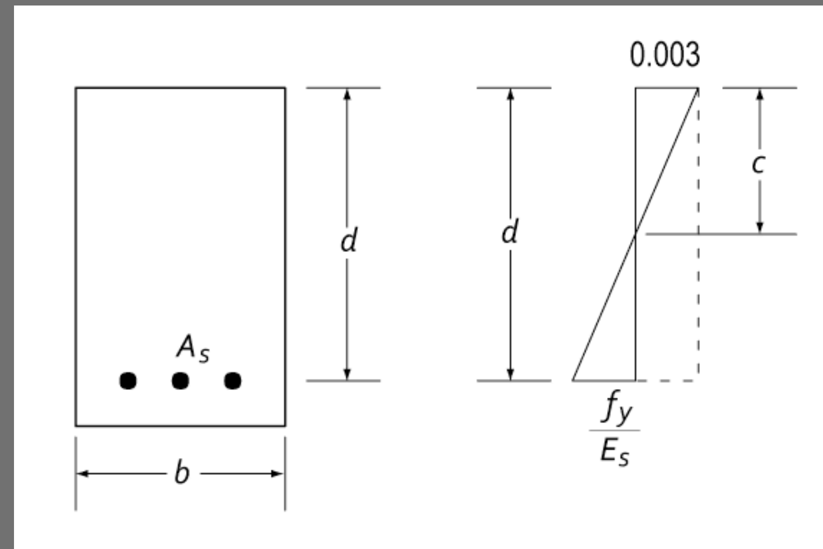
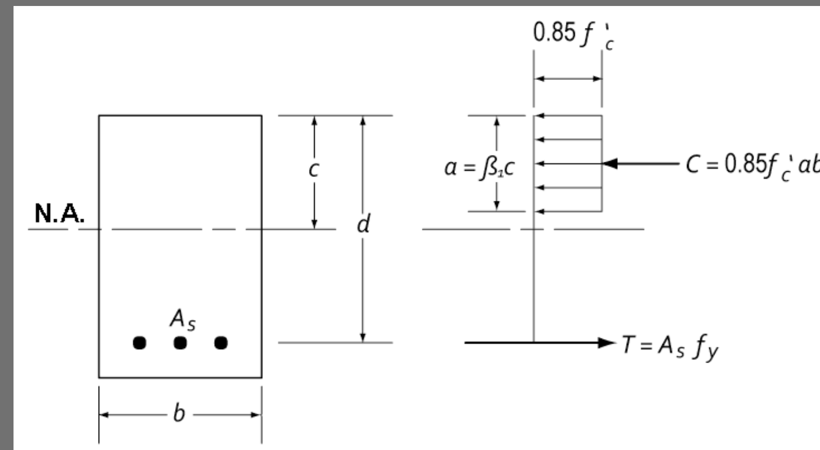


Table A.8 Balanced Ratio of Reinforcement ρ_b for Rectangular Sections with Tension Reinforcement Only

f_y \ f_c'		2,500 psi (17.2 MPa) $\beta_1 = 0.85$	3,000 psi (20.7 MPa) $\beta_1 = 0.85$	4,000 psi (27.6 MPa) $\beta_1 = 0.85$	5,000 psi (34.5 MPa) $\beta_1 = 0.80$	6,000 psi (41.4 MPa) $\beta_1 = 0.75$
Grade 40	ρ_b	0.0309	0.0371	0.0495	0.0582	0.0655
40,000 psi	$0.75\rho_b$	0.0232	0.0278	0.0371	0.0437	0.0492
(275.8 MPa)	$0.50\rho_b$	0.0155	0.0186	0.0247	0.0291	0.0328
Grade 50	ρ_b	0.0229	0.0275	0.0367	0.0432	0.0486
50,000 psi	$0.75\rho_b$	0.0172	0.0206	0.0275	0.0324	0.0365
(344.8 MPa)	$0.50\rho_b$	0.0115	0.0138	0.0184	0.0216	0.0243
Grade 60	ρ_b	0.0178	0.0214	0.0285	0.0335	0.0377
60,000 psi	$0.75\rho_b$	0.0134	0.0161	0.0214	0.0252	0.0283
(413.7 MPa)	$0.50\rho_b$	0.0089	0.0107	0.0143	0.0168	0.0189
Grade 75	ρ_b	0.0129	0.0155	0.0207	0.0243	0.0274
75,000 psi	$0.75\rho_b$	0.0097	0.0116	0.0155	0.0182	0.0205
(517.1 MPa)	$0.50\rho_b$	0.0065	0.0078	0.0104	0.0122	0.0137

Image Sources: University of Michigan, Department of Architecture

Rectangular Beam Analysis



Data:

- Section dimensions – b, h, d, (span)
- Steel area - As
- Material properties – f'c, fy

Required:

- Strength (of beam) Moment - Mn
- Required (by load) Moment – Mu
- Load capacity

$$a = \frac{A_s f_y}{0.85 f_c' b} \; or \; \frac{\rho f_y d}{0.85 f_c'}$$

1. Find ρ = As/bd

   (check ρ min< ρ < ρ max)

2. Find a
3. Find Mn
4. Calculate Mu<= ϕ Mn
5. Determine max. loading (or span)

$$M_n = A_s f_y \left(d - \frac{a}{2} \right)$$

$$M_u \leq \phi M_n$$

$$M_u = \frac{(1.4 w_{DL} + 1.7 w_{LL}) l^2}{8}$$

$$1.7 w_{LL} = \frac{M_u 8}{l^2} - 1.4 w_{DL}$$

Image Sources: University of Michigan, Department of Architecture

Rectangular Beam Analysis

Data:

- dimensions – b, h, d, (span)
- Steel area - As
- Material properties – f'c, fy

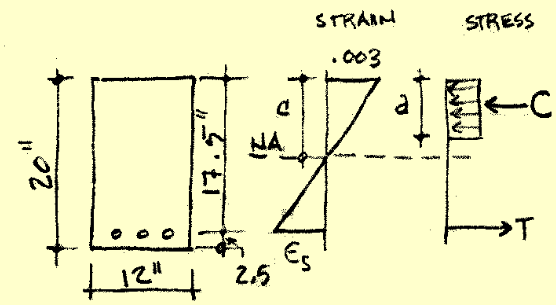
Required:

- Required Moment – Mu

1. Find ρ = As/bd
   (check ρ min< ρ < ρ max)

$f'_c = 4000$ psi
$f_y = 60000$ psi

REINF. = $A_s = 3 \times \#8$



FIND M_u, CHECK ρ

$$\rho = \frac{A_s}{bd} = \frac{2.37}{12(17.5)} = 0.0113$$

CHECK ρ:

ρ BALANCE

$$\rho_b = \left(\frac{.85 f'_c \beta_1}{f_y}\right)\left(\frac{87000}{87000 + f_y}\right)$$

$$= \left(\frac{.85(4)(.85)}{60}\right)\left(\frac{87000}{87000+60000}\right)$$

$$= 0.0285$$

$$\rho_{max} = 0.75\rho_b = 0.75(0.0285) = 0.0214$$

$$\rho_{min} = \frac{200}{f_y} = \frac{200}{60000} = 0.0033$$

$$0.0214 > 0.0113 > 0.0033 \quad \checkmark \text{OK}$$

Rectangular Beam Analysis cont.

2. Find a

$$a = \frac{A_s f_y}{.85 f'_c b} = \frac{(2.37)(60000)}{.85(4000)(12)} = 3.49$$

3. Find Mn

$$M_n = A_s f_y \left(d - \frac{a}{2}\right)$$

4. Find Mu

$$M_u = \phi A_s f_y \left(d - \frac{a}{2}\right)$$

$$M_u = .9(2.37)(60000)\left(17.5 - \frac{3.49}{2}\right)$$

$$M_u = 2017000 \text{ in-lb}$$

$$\underline{M_u = 168 \text{ ft-k}}$$

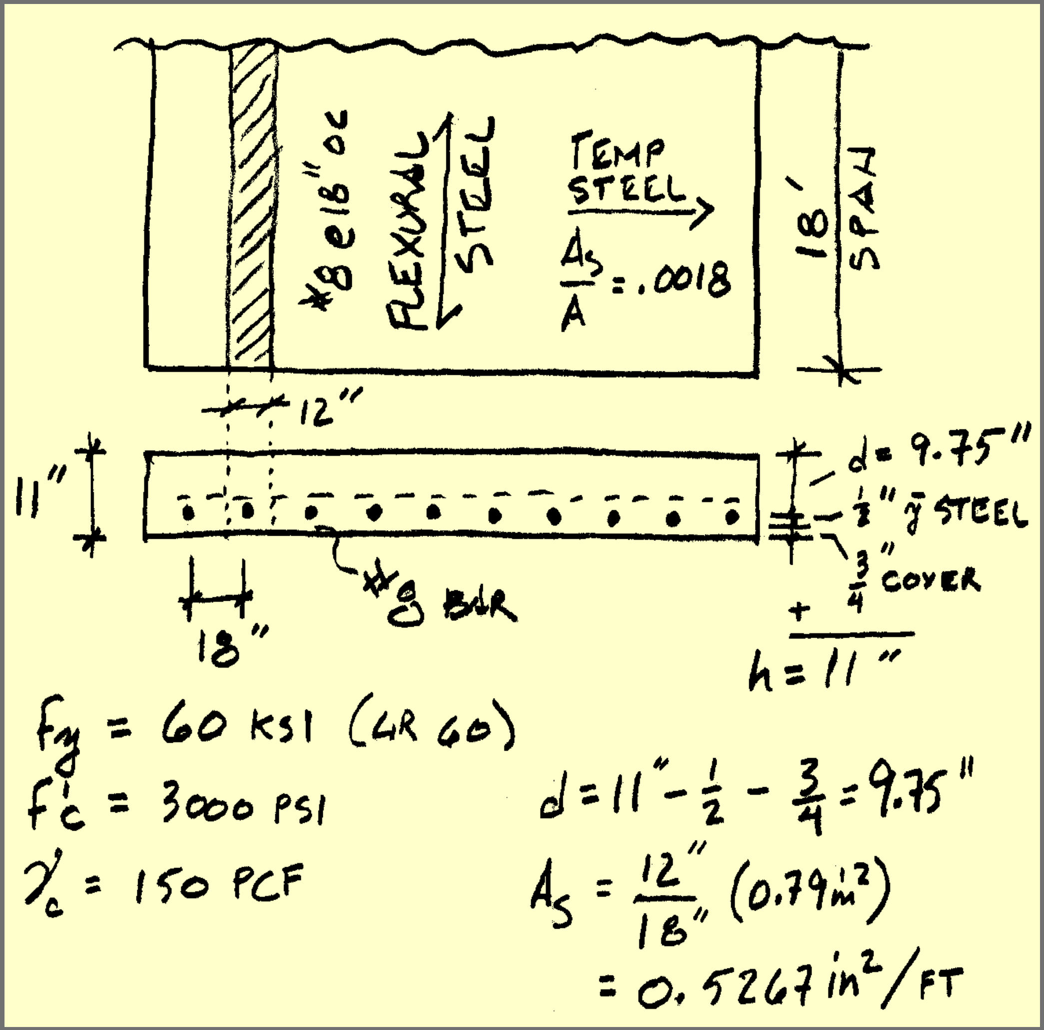
Slab Analysis

Data:

- Section dimensions – h, span
  take b = 12”
- Steel area - As
- Material properties – f’c, fy

Required:

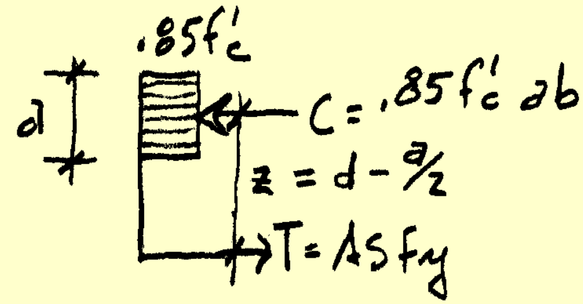
- Required Moment – Mu
- Maximum LL in PSF



Slab Analysis

1. Find a
2. Find force T
3. Find moment arm z
4. Find strength moment Mn

$$a = \frac{A_s f_y}{.85 f'_c b} = \frac{0.5267(60)}{.85(3)(12)} = 1.033''$$



$$T = A_s f_y = 0.5267(60) = 31.6^{K}$$

$$z = d - a/2 = 9.75 - \frac{1.033}{2} = 9.23''$$

$$M_n = Tz = 31.6(9.23) = 291.8^{K\text{-}''}$$

$$= 24317^{'\text{-}\#}$$

Slab Analysis

5. Find slab DL
6. Find Mu
7. Determine max. loading

$$w_{DL} = \gamma_c \frac{AREA_{in^2}}{144} = 150 \frac{11(12)}{144} = 137.5 \text{ PSF}$$

$$w_{U_{DL}} = 1.4(w_{DL}) = 1.4(137.5) = 192.5$$

$$w_{U_{LL}} = 1.7(w_{LL})$$

$$M_U = \frac{(w_{U_{DL}} + w_{U_{LL}})\,\ell^2}{8} = \phi M_n$$

$$0.9(24317) = \frac{[192.5 + 1.7(w_{LL})]\,18^2}{8}$$

$$w_{LL} = 204.6 \text{ PSF}$$

Rectangular Beam Design

Data:
- Load and Span
- Material properties – f'c, fy
- All section dimensions – b and h

Required:
- Steel area - As

1. Calculate the dead load and find Mu
2. d = h – cover – stirrup – $d_b/2$ (one layer)
3. Estimate moment arm jd (or z) ≅ 0.9 d and find As
4. Use As to find a
5. Use a to find As (repeat…)
6. Choose bars for As and check ρ max & min
7. Check Mu< ϕ Mn (final condition)
8. Design shear reinforcement (stirrups)
9. Check deflection, crack control, steel development length.

$$M_u = \frac{(1.4w_{DL} + 1.7w_{LL})l^2}{8}$$

$$A_s = \frac{M_u}{\phi\, f_y\left(d - \frac{a}{2}\right)}$$

$$a = \frac{A_s f_y}{0.85 f_c' b}$$

$$M_n = A_s f_y\left(d - \frac{a}{2}\right)$$

Rectangular Slab Design

DATA: ONE-WAY FLOOR SLAB – SPAN = 18 FEET

$f_y = 60000$ psi

$f'_c = 3000$ psi

$\gamma_{conc} = 150\,\#/f^3$

$\rho = \frac{1}{2}\rho_{max} = .008$

LL = 200 psf

DL = SLAB WEIGHT

REQUIRED: h AND A_s

Data:

- Load and Span
- Material properties – f'c, fy

Required:

- All section dimensions – h
- Steel area - As

1. Calculate the dead load and find Mu
2. Estimate moment arm jd (or z) ≅ 0.9 d and find As
3. Use As to find a
4. Use a to find As (repeat…)

ASSUME h

$h = \ell/20 = 10.8$ USE 11"

CALCULATE LOADS

DL = 137.5 lb/f²

LL = 200.0 lb/f²

$w = 1.4(137) + 1.7(200) = 540$

CALCULATE Mu

$$M_u = \frac{w\ell^2}{8} = 21.7^{K\text{-}f}$$

INITIAL As TRIAL

USE $.9d = .9 \times (11 - .75 - .25) = 9"$

$$A_s = \frac{M_u}{\phi f_y (d - \frac{a}{2})} \approx \frac{21.7 \times 12}{.9(60)(9)} = .536$$

INITIAL a

$$a = \frac{A_s f_y}{.85 f'_c b} = \frac{.536(60)}{.85(3)(12)} = 1.05"$$

Rectangular Slab Design

3. Use As to find a
4. Use a to find As (repeat…)
5. Choose bars for As and check As min & As max
6. Check Mu<ϕ Mn (final condition)
7. Check deflection, crack control, steel development length.

As

$$A_s = \frac{M_u}{\phi f_y (d - \frac{a}{2})} = \frac{21.7 \times 12}{.9(60)(10 - \frac{1.05}{2})} = .508$$

a

$$a = \frac{.508(60)}{.85(3)(12)} = .998$$

As

$$A_s = \frac{21.7 \times 12}{.9(60)(10 - \frac{.998}{2})} = .507 \text{ in} \approx .508 \checkmark$$

BAR SIZE + SPACING

using #4 bar

$\frac{.507}{12''} : \frac{.20}{s''}$ s = 4.7″   ∴ USE #4 @ 4″ o.c.

alternate - for max. s of 18″

$\frac{.507}{12''} : \frac{w_r}{18''} \rightarrow .761 \approx$ #8 @ .79   ∴ USE #8 @ 18″ o.c.

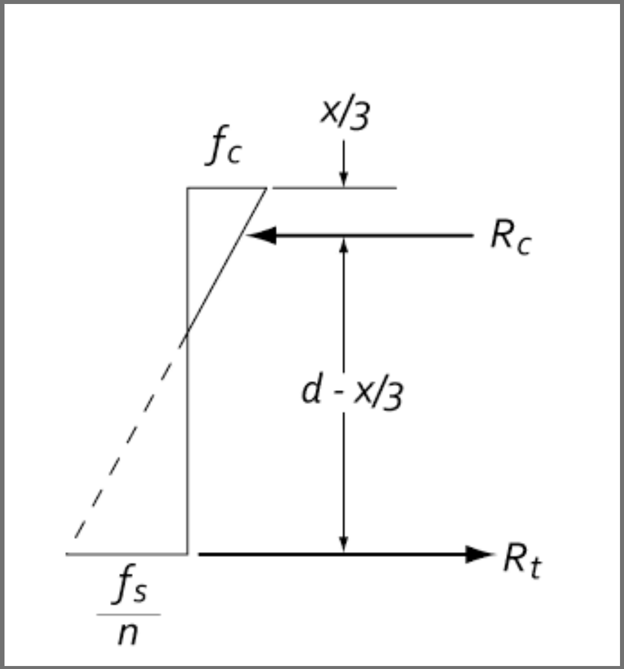
$A_{s\,min} = .0018\,bh = .24 < .5$ ✓ok

Quiz 9

Can f = Mc/I be used in Ult. Strength concrete beam calculations? (yes or no)

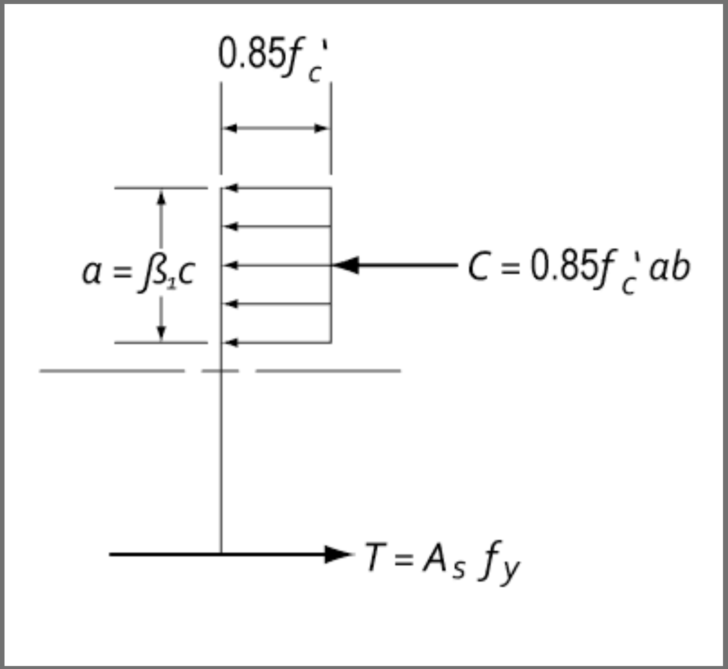
HINT:

WSD stress



Source: University of Michigan, Department of Architecture

Ult. Strength stress



Source: University of Michigan, Department of Architecture

Rectangular Beam Design

Data:

- Load and Span
- Some section dimensions – b or d
- Material properties – f'c, fy

Required:

- Steel area - As
- Beam dimensions – b or d

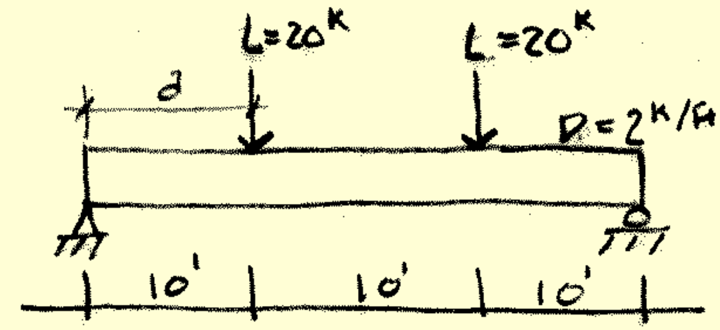
$$M_u = \frac{(1.4w_{DL} + 1.7w_{LL})l^2}{8}$$

$$bd^2 = \frac{M_u}{\phi\rho f_y\left(1 - 0.59\rho\left(f_y / f'_c\right)\right)}$$

$$A_s = \rho bd$$

1. Choose ρ (e.g. 0.5 ρ max or 0.18f'c/fy)
2. Estimate the dead load and find Mu
3. Calculate bd^2
4. Choose b and solve for d
   b is based on form size – try several to find best
5. Estimate h and correct weight and Mu
6. Find As= ρ bd
7. Choose bars for As and determine spacing and cover. Recheck h and weight.
8. Design shear reinforcement (stirrups)
9. Check deflection, crack control, steel development length.

Rectangular Beam Design



$f'_c = 3000$ psi

$f_y = 60000$ psi

Data:

- Load and Span
- Material properties – f'c, fy

Required:

- Steel area - As
- Beam dimensions – b and d

1. Estimate the dead load and find Mu
2. Choose ρ (e.g. 0.5 ρ max or 0.18f'c/fy)

FACTORED LL = P_L = 1.7(L) = 1.7(20) = 34 K

FACTORED DL = W_D = 1.4 (APPLIED LOAD + BEAM WEIGHT ESTIMATE) = 1.4(2+.6) = 3.64 K/ft

$$M_u = P_L a + \frac{w_D \ell^2}{8} = 34(10) + \frac{3.64 \times 30^2}{8} = 340 + 409.5 = 749.5 \text{ k-ft}$$

M_u = 8994000 in-lb

$$\rho = \frac{.18 f'_c}{f_y} = .009$$

Rectangular Beam Design cont

3. Calculate bd^2

$$bd^2 = \frac{M_u}{\phi \rho f_y (1 - 0.59 \rho (f_y / f'_c))}$$

$$bd^2 = \frac{8994}{(.9)(.009)(60)(1 - .59(.009)(60/3))}$$

$$bd^2 = 20705 \text{ in}^3$$

4. Choose b and solve for d
   b is based on form size.
   try several to find best

POSSIBILITIES

b	x	d
14"		38.5"
16"		35.97"
18"		33.9"

Rectangular Beam Design

5. Estimate h and correct weight and Mu
6. Find As= ρ bd
7. Choose bars for As and determine spacing and cover. Recheck h and weight.
8. Design shear reinforcement (stirrups)
9. Check deflection, crack control, steel development length.

CHECK BEAM WEIGHT:

$\frac{37 \times 18}{144} \times 150 = 694$ ✓ 2% ERROR OK ✓

(SEE BELOW)

$\rho = .009 = \frac{A_s}{bd}$, $A_s = .009\ bd = .009 \times 18 \times 34$

$A_s = 5.5\ in^2$

TABLE A-4

USE 7 x #8^s

SPACED WITH 1" BETWEEN EACH BAR

Table A.4 Areas of Groups of StandardBars (in.^{2})

Bar No.	Number of Bars 2	3	4	5	6	7	8	9	10	11	12	13	14
4	0.39	0.58	0.78	0.98	1.18	1.37	1.57	1.77	1.96	2.16	2.36	2.55	2.75
5	0.61	0.91	1.23	1.53	1.84	2.15	2.45	2.76	3.07	3.37	3.68	3.99	4.30
6	0.88	1.32	1.77	2.21	2.65	3.09	3.53	3.98	4.42	4.86	5.30	5.74	6.19
7	1.20	1.80	2.41	3.01	3.61	4.21	4.81	5.41	6.01	6.61	7.22	7.82	8.42
8	1.57	2.35	3.14	3.93	4.71	5.50	6.28	7.07	7.85	8.64	9.43	10.21	11.00
9	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00	11.00	12.00	13.00	14.00
10	2.53	3.79	5.06	6.33	7.59	8.86	10.12	11.39	12.66	13.92	15.19	16.45	17.72
11	3.12	4.68	6.25	7.81	9.37	10.94	12.50	14.06	15.62	17.19	18.75	20.31	21.87
14	4.50	6.75	9.00	11.25	13.50	15.75	18.00	20.25	22.50	24.75	27.00	29.25	31.50
18	8.00	12.00	16.00	20.00	24.00	28.00	32.00	36.00	40.00	44.00	48.00	52.00	56.00

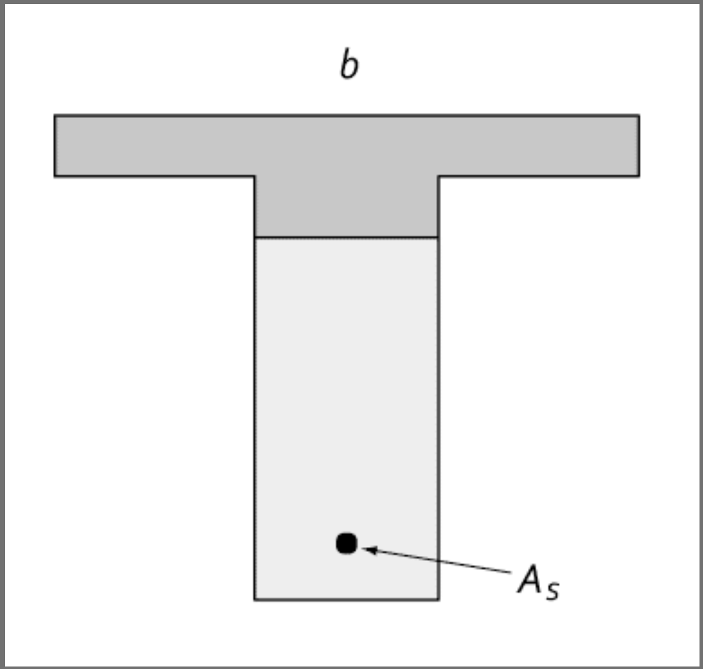
Source: Jack C McCormac, 1978 Design of Reinforced Concrete, Harper and Row, 1978

Non-Rectangular Beam Analysis

Data:
- Section dimensions – b, h, d, (span)
- Steel area - As
- Material properties – f'c, fy

Required:
- Required Moment – Mu (or load, or span)

1. Draw and label diagrams for section and stress
    1. Determing b effective (for T-beams)
    2. Locate T and C (or C_1 and C_2)
2. Set T=C and write force equations (P=FA)
    1. T = As fy
    2. C = 0.85 f'c Ac
3. Determine the Ac required for C
4. Working from the top down, add up area to make Ac
5. Find moment arms (z) for each block of area
6. Find Mn = Σ Cz
7. Find Mu = ϕ Mn ϕ =0.90
8. Check As min < As < As max

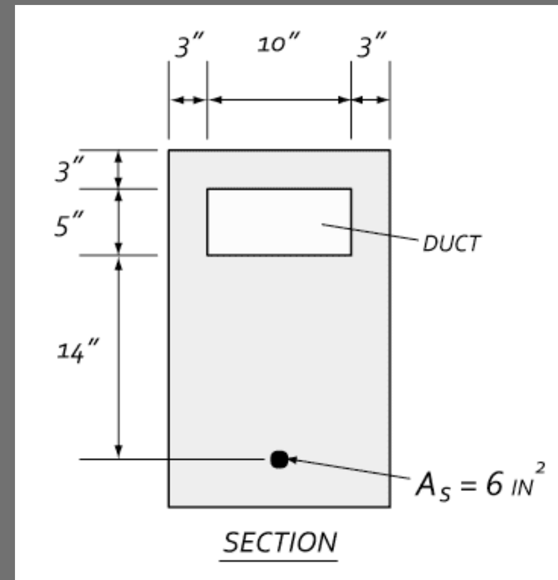


Source: University of Michigan, Department of Architecture

Analysis Example

Given: $f'c = 3000$ psi
$fy = 60$ ksi
$As = 6\ in^2$

Req'd: Capacity, Mu



Source: University of Michigan, Department of Architecture

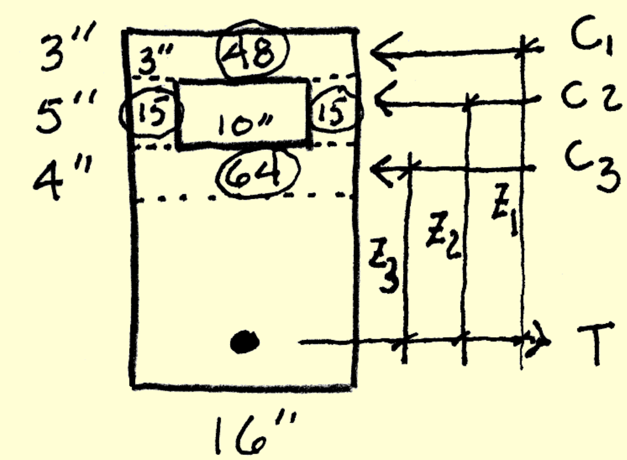
1. Find T
2. Find C in terms of Ac
3. Set T=C and solve for Ac

$$T = A_s f_y = 6\,in^2\,(60000\,psi)$$
$$T = 360\,000^{\#} = 360^{k}$$
$$C = 0.85 f'_c A_c = 0.85(3000\,psi)\,A_c\,in^2$$
$$C = (2550\,A_c)^{\#} = (2.55\,A_c)^{k}$$
$$T = C$$
$$360^{k} = 2.55\,A_c{}^{k}$$
$$A_c = 142\,in^2$$

Example

4. Draw section and determine areas to make Ac

5. Solve C for each area in compression.



$$A_c = 142\ in^2 = A_{c1} + A_{c2} + A_{c3}$$

$$142 = 48 + 30 + A_{c3}$$

$$A_{c3} = 64\ in^2$$

$$C_1 = 48(2.55) = 122.4^k$$

$$C_2 = 30(2.55) = 76.5^k$$

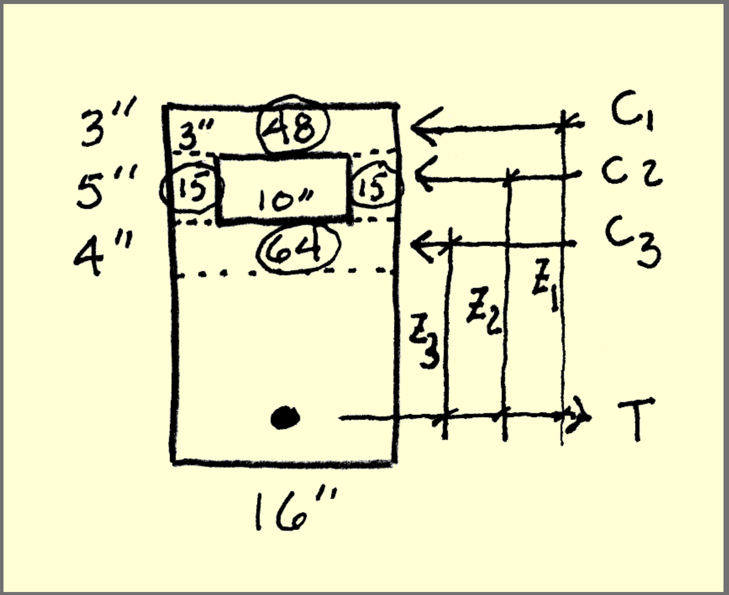
$$C_3 = 64(2.55) = 163.2^k$$

Example

6. Determine moment arms to areas, z.

7. Calculate Mn by summing the Cz moments.

8. Find Mu = φMn



$z_1 = 22 - 1.5 = 20.5''$

$z_2 = 22 - (3 + 2.5) = 16.5''$

$z_3 = 22 - (8 + 2) = 12.0''$

$M_n = \Sigma C z$

$M_n = (C_1 z_1) + (C_2 z_2) + (C_3 z_3)$

$M_n = 2509 + 1262 + 1959$

$M_n = 5730$

$M_u = \phi M_n = 0.9(5730) = 5157^{k\text{-}'}$

Other Useful Tables:

Table A.1 Values of Modulus of Elasticity for Normal-Weight Concrete

Customary Units		SI Units	
f_c' (psi)	E_c (psi)	f_c' (MPa)	E_c (MPa)
3,000	3,140,000	20.7	21 650
3,500	3,390,000	24.1	23 373
4,000	3,620,000	27.6	24 959
4,500	3,850,000	31.0	26 545
5,000	4,050,000	34.5	27 924

Table A.2 Designations, Areas, Perimeters, and Weights of Standard Bars

	Customary Units			SI Units		
Bar No.	Diameter (in.)	Cross-sectional Area (in.^{2})	Unit Weight (lb/ft)	Diameter (mm)	Cross-sectional Area (mm^2)	Unit Weight (kg/m)
3	0.375	0.11	0.376	9.52	71	0.560
4	0.500	0.20	0.668	12.70	129	0.994
5	0.625	0.31	1.043	15.88	200	1.552
6	0.750	0.44	1.502	19.05	284	2.235
7	0.875	0.60	2.044	22.22	387	3.042
8	1.000	0.79	2.670	25.40	510	3.973
9	1.128	1.00	3.400	28.65	645	5.060
10	1.270	1.27	4.303	32.26	819	6.404
11	1.410	1.56	5.313	35.81	1006	7.907
14	1.693	2.25	7.650	43.00	1452	11.384
18	2.257	4.00	13.600	57.33	2581	20.238

Table A.4 Areas of Groups of StandardBars (in.^{2})

	Number of Bars												
Bar No.	2	3	4	5	6	7	8	9	10	11	12	13	14
4	0.39	0.58	0.78	0.98	1.18	1.37	1.57	1.77	1.96	2.16	2.36	2.55	2.75
5	0.61	0.91	1.23	1.53	1.84	2.15	2.45	2.76	3.07	3.37	3.68	3.99	4.30
6	0.88	1.32	1.77	2.21	2.65	3.09	3.53	3.98	4.42	4.86	5.30	5.74	6.19
7	1.20	1.80	2.41	3.01	3.61	4.21	4.81	5.41	6.01	6.61	7.22	7.82	8.42
8	1.57	2.35	3.14	3.93	4.71	5.50	6.28	7.07	7.85	8.64	9.43	10.21	11.00
9	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00	11.00	12.00	13.00	14.00
10	2.53	3.79	5.06	6.33	7.59	8.86	10.12	11.39	12.66	13.92	15.19	16.45	17.72
11	3.12	4.68	6.25	7.81	9.37	10.94	12.50	14.06	15.62	17.19	18.75	20.31	21.87
14	4.50	6.75	9.00	11.25	13.50	15.75	18.00	20.25	22.50	24.75	27.00	29.25	31.50
18	8.00	12.00	16.00	20.00	24.00	28.00	32.00	36.00	40.00	44.00	48.00	52.00	56.00

Image Sources: Jack C McCormac, 1978 Design of Reinforced Concrete, Harper and Row, 1978