# Continuum Physics

**Midterm Exam**

**Problem 1.**

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$ be three, linearly independent vectors. Let $\boldsymbol{A} \in \mathbb{GL}(3)$ be the tensor given by $\boldsymbol{A} = \boldsymbol{a} \otimes (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \otimes (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \otimes (\boldsymbol{a} \times \boldsymbol{b})$. Denote the scalar triple product of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ by $V = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$. Finally, define the unit vectors

$$\boldsymbol{a}^* = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}, \quad \boldsymbol{b}^* = \frac{\boldsymbol{b}}{|\boldsymbol{b}|}, \boldsymbol{c}^* = \frac{\boldsymbol{c}}{|\boldsymbol{c}|}. \tag{1}$$

It can be shown that

$$\boldsymbol{a}^* \cdot \boldsymbol{A}\boldsymbol{a}^* = V, \quad \boldsymbol{b}^* \cdot \boldsymbol{A}\boldsymbol{b}^* = V, \quad \boldsymbol{c}^* \cdot \boldsymbol{A}\boldsymbol{c}^* = V, \tag{2}$$

$$\boldsymbol{a}^* \cdot \boldsymbol{A}\boldsymbol{b}^* = \boldsymbol{b}^* \cdot \boldsymbol{A}\boldsymbol{a}^* = V\boldsymbol{a}^* \cdot \boldsymbol{b}^*, \quad \boldsymbol{b}^* \cdot \boldsymbol{A}\boldsymbol{c}^* = \boldsymbol{c}^* \cdot \boldsymbol{A}\boldsymbol{b}^* = V\boldsymbol{b}^* \cdot \boldsymbol{c}^*, \quad \boldsymbol{c}^* \cdot \boldsymbol{A}\boldsymbol{a}^* = \boldsymbol{a}^* \cdot \boldsymbol{A}\boldsymbol{c}^* = V\boldsymbol{c}^* \cdot \boldsymbol{a}^* \tag{3}$$

Using these results show that $\boldsymbol{A} = V\mathbf{1}$, where $\mathbf{1}$ is the usual second-order isotropic tensor.

**Note**: There is a smarter way to do this than writing out vector/tensor components explicitly.

**Problem 2.**

Consider a body, which in the reference configuration, $\Omega_0$, is a sphere of radius $R$. Its deformation gradient is $\boldsymbol{F}$.

(a) If $\boldsymbol{F} = \lambda\mathbf{1}$, where $\mathbf{1}$ is the usual second-order isotropic tensor, what is the shape of the body in its deformed configuration, $\Omega_t$?

(b) If $\boldsymbol{F} = \lambda_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \lambda_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \lambda_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$, where $\lambda_1 \neq \lambda_2 \neq \lambda_3$ are constants and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a constant orthonormal basis, then what is the shape of the deformed configuration, $\Omega_t$? Provide an explicit parametrization for it.

**Problem 3.**

Let $\Omega$ be the current configuration of a body. The heat flux vector (heat crossing a unit area in the current configuration per unit time in the direction perpendicular to the area) is $\boldsymbol{q} \in \mathbb{R}^3$. The heat supply (heat supplied externally per unit current volume per unit time) is $r \in \mathbb{R}$, a scalar. The internal energy per unit mass is $e \in \mathbb{R}$ a scalar. The mass density is $\rho$. The heat entering $\Omega$ through its boundary, $\partial\Omega$ is $h = -\boldsymbol{q} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the unit outward normal.

(a) Derive the *local* balance of energy equation which corresponds to the following specialization of the First Law of Thermodynamics: "The rate of change of internal energy is equal to the net heat supplied to the body by external heating at every point, and through the boundary."

(b) Consider Fig 1, where the current configuration $\Omega$ is shown as the union of two *open, disjoint* subsets, $\Omega_1$ and $\Omega_2$. That is, $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$ and $\Omega_1 \cap \Omega_2 = \emptyset$. The interface between $\Omega_1$ and $\Omega_2$ is $\Gamma$, with unit normal $\boldsymbol{m}$ *pointing into* $\Omega_2$ as shown. So, the boundary of $\Omega_1$ is $\Gamma_1 \cup \Gamma$ and the boundary of $\Omega_2$ is $\Gamma_2 \cup \Gamma$.

Apply the local form of the balance of energy equation that you derived in Part (a) to each open subset, $\Omega_1$ and $\Omega_2$. Then obtain the *integral* form of the balance of energy for this problem. Given that there is no heat loss across the interface, $\Gamma$, what can you say about the heat flux vector across $\Gamma$?

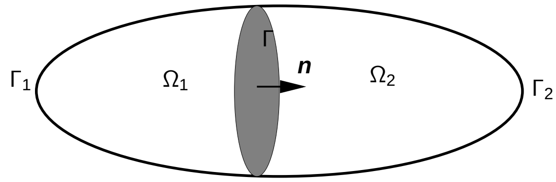


Figure 1: The body in its current configuration is the union of subsets: $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$ and $\Omega_1 \cap \Omega_2 = \emptyset$.

**Problem 4.**

The Kirchhoff stress tensor is defined as $\boldsymbol{\tau} = J\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the Cauchy stress and $J = \det\boldsymbol{F}$ is the volume change ratio written in terms of the deformation gradient, $\boldsymbol{F}$. The left Cauchy-Green tensor, $\boldsymbol{b}$, satisfies $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$.

(a) For an *isotropic, hyperelastic* solid with strain energy density $\psi$ show that

$$\boldsymbol{\tau} = 2\frac{\partial\psi}{\partial\boldsymbol{b}}\boldsymbol{b}.$$

(b) For an *isotropic, hyperelastic* solid with strain energy density $\psi$, how are the principal values of the Kirchhoff stress tensor related to $\psi$ and the principal stretches, $\lambda_1, \lambda_2, \lambda_3$?