<u>Continuum Physics</u>

Midterm Exam

<u>Problem 1.</u>

Consider a rigid body motion expressed as $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t) = \boldsymbol{g}(t) + \boldsymbol{Q}(t)(\boldsymbol{X} - \boldsymbol{G})$, where $\boldsymbol{x}, \boldsymbol{X}$ are the current and reference positions of a point, $\boldsymbol{\varphi}$ is the motion, $\boldsymbol{g}, \boldsymbol{G}$ are the current and reference positions of the center of mass, and $\boldsymbol{Q} \in \mathrm{SO}(3)$ is the rotation tensor. Let the total mass of the body be m, the angular velocity be $\hat{\boldsymbol{\omega}}(t)$, and let the angular momentum about the origin be $\mathscr{J}_0$. We showed in class that $\mathscr{J}_0$ can be written as

$$\mathscr{J}_0 = \boldsymbol{g} \times m\dot{\boldsymbol{g}} + \boldsymbol{I}_g \hat{\boldsymbol{\omega}} \tag{1}$$

where

$$\boldsymbol{I}_g(t) := \int_{\Omega_t} \left[|\boldsymbol{x} - \boldsymbol{g}|^2 \mathbf{1} - (\boldsymbol{x} - \boldsymbol{g}) \otimes (\boldsymbol{x} - \boldsymbol{g}) \right] \rho \mathrm{d}v. \tag{2}$$

Also define for convenience

$$\boldsymbol{E}_g(t) := \int_{\Omega_t} (\boldsymbol{x} - \boldsymbol{g}) \otimes (\boldsymbol{x} - \boldsymbol{g}) \rho \mathrm{d}v. \tag{3}$$

Show that the rate of change of angular momentum is

$$\dot{\mathscr{J}}_0 = \boldsymbol{g} \times m\ddot{\boldsymbol{g}} + \boldsymbol{I}_g \dot{\hat{\boldsymbol{\omega}}} - \hat{\boldsymbol{\omega}} \times (\boldsymbol{E}_g \hat{\boldsymbol{\omega}}).$$

Problem 2.

Let Ω be the current configuration of a body. It is loaded by a body force per unit current volume, $\boldsymbol{f}$, and the surface traction vector is $\boldsymbol{t}$. It has spatial velocity, $\boldsymbol{v}$, varying pointwise. You may denote the Cauchy stress as $\boldsymbol{\sigma}$ if you need it. Additionally, the heat flux vector (heat crossing a unit area in the current configuration per unit time in the direction perpendicular to the area) is $\boldsymbol{q} \in \mathbb{R}^3$. The heat supply (heat supplied externally per unit current volume per unit time) is $r \in \mathbb{R}$, a scalar. The internal energy per unit mass is $e \in \mathbb{R}$, a scalar. The mass density is ρ. The heat entering Ω through its boundary, $\partial\Omega$ is $h = -\boldsymbol{q} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the unit outward normal. The total energy is the internal energy plus the kinetic energy.

Derive the *local* balance of energy equation which corresponds to the following statement of the First Law of Thermodynamics: "*The rate of change of total energy is equal to the sum of (i) the net rate of heat supplied to the body by external heating at every point, and through the boundary, and (ii) the rate of work done on the body.*"

Problem 3.

Which of the following constitutive equations for the Cauchy stress, $\boldsymbol{\sigma}$, transform according to the rules for frame invariance under rigid body motions of the current configuration, and which do not? Justify or qualify your responses as necessary. Here, α is a scalar constant, and $\boldsymbol{f}$ is a symmetric tensor-valued function. Besides these, $\boldsymbol{F}$ is the deformation gradient tensor, $\boldsymbol{v}$ is the spatial velocity, $\boldsymbol{a}$ is the spatial acceleration, and $\boldsymbol{l}$ is the spatial velocity gradient tensor ($\nabla \boldsymbol{v}$).

(a) $\boldsymbol{\sigma} = \alpha(\boldsymbol{F} + \boldsymbol{F}^{\mathrm{T}})$

(b) $\boldsymbol{\sigma} = \boldsymbol{f}(\boldsymbol{v})$

(c) $\boldsymbol{\sigma} = \alpha[\nabla \boldsymbol{a} + \nabla \boldsymbol{a}^{\mathrm{T}} + 2\boldsymbol{l}^{\mathrm{T}}\boldsymbol{l}]$

Problem 4.

Consider the quadratic-logarithmic strain energy density function, expressed as a function of principal stretches, $\lambda_1, \lambda_2, \lambda_3$, and with material Lamé parameters λ and μ. (*Do not confuse* λ with $\lambda_1, \lambda_2, \lambda_3$!)

$$\hat{W}(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{2}\lambda \left(\log \lambda_1 \lambda_2 \lambda_3\right)^2 + \mu \left((\log \lambda_1)^2 + (\log \lambda_2)^2 + (\log \lambda_3)^2 \right). \tag{4}$$

For simplicity we will take the Cartesian basis vectors and the principal stretch directions to coincide. So, the principal directions corresponding to $\lambda_1, \lambda_2, \lambda_3$ are $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, rspectively. With this simplification show that the above strain energy density function reduces to the the strain energy density function used in linearized elasticity, and defined in terms of the infinitesimal normal strains $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$.