Continuum Physics

Final homework

Problem 1.

Which of the following constitutive equations for the Cauchy stress, $\boldsymbol{\sigma}$, are objective (transform correctly under rigid body motions of the current configuration) and which are not? Justify your responses. Here, α and β are scalar constants, p is a scalar-valued function and $\boldsymbol{f}$ is a symmetric tensor-valued function. Besides these terms, $\boldsymbol{F}$ is the deformation gradient tensor, $\boldsymbol{v}$ is the spatial velocity, $\boldsymbol{a}$ is the spatial acceleration, $\boldsymbol{l}$ is the spatial velocity gradient tensor $(\nabla \boldsymbol{v})$, $\boldsymbol{W} = \frac{1}{2}(\boldsymbol{l} - \boldsymbol{l}^{\mathrm{T}})$ is the spin tensor, $\boldsymbol{d} = \frac{1}{2}(\boldsymbol{l} + \boldsymbol{l}^{\mathrm{T}})$ is the rate of deformation tensor, and $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ is the left Cauchy-Green tensor.

(a) $\boldsymbol{\sigma} = -p(t)\mathbf{1}$

(b) $\boldsymbol{\sigma} = \alpha(\boldsymbol{F} + \boldsymbol{F}^{\mathrm{T}})$

(c) $\boldsymbol{\sigma} = \boldsymbol{f}(\boldsymbol{v})$

(d) $\boldsymbol{\sigma} = \alpha[\nabla \boldsymbol{a} + \nabla \boldsymbol{a}^{\mathrm{T}} + 2\boldsymbol{l}^{\mathrm{T}}\boldsymbol{l}]$

(e) $\boldsymbol{\sigma} = \boldsymbol{f}(\boldsymbol{b})$

(f) $\dot{\boldsymbol{\sigma}} = \boldsymbol{W}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{W} + \alpha \mathrm{tr}(\boldsymbol{d})\mathbf{1} + \beta \boldsymbol{d}$

Problem 2.

Consider the steady flow of an incompressible viscous Newtonian fluid with zero body forces and a velocity field given $\boldsymbol{v}(\boldsymbol{x}) = v_1(x_1, x_2)\boldsymbol{e}_1$.

(a) Suppose that the fluid is flowing through two infinite flat plates, one at $x_2 = 0$ and one at $x_2 = h, > 0$, with the bottom plate held fixed and the top plate moving in the x_1 direction with speed v. What are the boundary conditions on the fluid? Assuming there is no pressure drop in the x_1 direction, solve for the velocity component $v_1(x_1, x_2)$ and the stress tensor $\boldsymbol{\sigma}$.

(b) Consider the same configuration as part (a), but with both plates held stationary. What are the relevant boundary conditions? Assuming now that the pressure is *not* constant, solve for the velocity component $v_1(x_1, x_2)$ and the stress tensor $\boldsymbol{\sigma}$.

Problem 3.

Consider the *uniform extension* defined by

$$
\begin{aligned}
x_1 &= \lambda X_1 \\
x_2 &= \omega X_2 \\
x_3 &= \omega X_3
\end{aligned}
$$

(a) Compute $\boldsymbol{F}$ and $\boldsymbol{b}$.

(b) Suppose that the material is *isotropic elastic*. Compute the stress tensor $\boldsymbol{\sigma}$.

(c) Assume that the material is a cube of length L in the reference configuration, and compute the tractions.

(d) How must the constants, λ and ω, be related if $\boldsymbol{\sigma}$ is a pure tension in the direction of $\boldsymbol{e}_1$; i.e., if $\boldsymbol{\sigma} = \sigma(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1)$.

Problem 4.

Consider a two-dimensional motion described by $\boldsymbol{F} = \boldsymbol{RU}$, where

$$F_{\alpha\beta} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$

with respect to the basis $\{\boldsymbol{e}_\alpha\}$, $\alpha = 1, 2$.

(a) Write down the characteristic polynomial of $\boldsymbol{U}$, then use the Cayley-Hamilton Theorem (the matrix substituted into its own characteristic polynomial gives the zero matrix) to show that $\boldsymbol{U}$ can be written as

$$\boldsymbol{U} = \frac{1}{i_1}(\boldsymbol{C} + i_2\mathbf{1})$$

where $i_1 = \mathrm{tr}(\boldsymbol{U})$, $i_2 = \det(\boldsymbol{U})$ and $\mathbf{1}$ is the 2D identity tensor.

(b) Show that the invariants of $\boldsymbol{U}$ (i_1, i_2) can be written in terms of the invariants of $\boldsymbol{C}$ (I_1, I_2):

$$i_1 = \sqrt{I_1 + 2i_2}, \quad i_2 = \sqrt{I_2} = J$$

(c) Conclude from parts (a) and (b) that $\boldsymbol{U}$ may be written *in closed form* as

$$\boldsymbol{U} = \frac{1}{\sqrt{I_1 + 2J}}(\boldsymbol{C} + J\mathbf{1}).$$

(d) Consider now the case of simple shear in which $\boldsymbol{F}$ is given by

$$F_{\alpha\beta} = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$$

with respect to the basis $\{\boldsymbol{e}_\alpha\}$.

(i) Use the previous results to compute $\boldsymbol{U}$.

(ii) Compute $\boldsymbol{R}$.

(iii) Compare your results with those of Homework assignment 3, Problem 4, in which the same polar decomposition was determined via the spectral decomposition of $\boldsymbol{C}$.

Problem 5.

Consider the torsion of an isotropic linear elastic prismatic beam. Recall the St. Venant assumption from class:

$$u_\alpha = -e_{\alpha\beta}\theta(x_3)x_\beta, \ \alpha, \beta = 1, 2; \quad u_3 = a\psi(x_1, x_2),$$

where $\theta(x_3) = ax_3$ and a is a constant.

(a) Show that the resultant force on any cross section $x_3 =$ constant is zero.

(b) Derive the expression for the resultant torque on the plane $x_3 = L/2$.

(c) Derive the following expression for the torsion modulus:

$$\mu \bar{J} = \mu J - \mu \int_{\mathcal{S}} \|\nabla\psi\|^2 \mathrm{d}x_1 \mathrm{d}x_2,$$

where J is the polar moment of inertia and μ is the shear modulus.

(d) Consider the case of a circular cross section. Find the solution of the problem:

$$\begin{aligned} \nabla^2\psi &= 0 \text{ in } \mathcal{S} \\ \frac{\partial\psi}{\partial n} &= e_{\alpha\beta} n_\alpha x_\beta \text{ on } \partial\mathcal{S}, \end{aligned}$$

where $\partial\psi/\partial n = \psi,_\alpha n_\alpha$. Assume that $\psi = \psi(r)$ where $r = \sqrt{x_1^2 + x_2^2}$ and $\nabla^2\psi = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\frac{\mathrm{d}\psi}{\mathrm{d}r})$.

Problem 6.

A rubber cylinder of radius A and length L in the reference configuration is rotating about its axis of symmetry with constant angular velocity ω. This motion is given by:

$$\begin{aligned} x_1 &= \lambda^{-\frac{1}{2}}(X_1 \cos\omega t - X_2 \sin\omega t) \\ x_2 &= \lambda^{-\frac{1}{2}}(X_1 \sin\omega t + X_2 \cos\omega t) \\ x_3 &= \lambda X_3, \end{aligned}$$

where λ is a positive constant. The rubber is incompressible and may be characterized by the equation

$$\boldsymbol{\sigma} = -p\mathbf{1} + [(\alpha + \beta(\mathrm{tr}(\boldsymbol{b}))]\boldsymbol{b} - \beta\boldsymbol{b}^2$$

where α and β are constants and $p = p(x_1, x_2, x_3, t)$ in general.

(a) Compute $\boldsymbol{v}$ and $\boldsymbol{a}$, the spatial velocity and acceleration.

(b) Compute the rate of deformation tensor, $\boldsymbol{d}(\boldsymbol{x}, t)$, the spin tensor $\boldsymbol{\omega}(\boldsymbol{x}, t)$ and the vorticity vector $\hat{\boldsymbol{\omega}}(\boldsymbol{x}, t)$.

(c) Verify that the motion is isochoric and determine the principal stretches.

(d) Assuming that there are no body forces, use the balance of linear momentum to derive the scalar relations for the pressure:

$$\frac{\partial p}{\partial x_1} = \rho\omega^2 x_1, \quad \frac{\partial p}{\partial x_2} = \rho\omega^2 x_2, \quad \frac{\partial p}{\partial x_3} = 0.$$

(e) Assuming that the curved boundary of the cylinder is traction-free, show that the pressure is given by

$$p = \frac{1}{2}\rho\omega^2(r^2 - \lambda^{-1}A^2) + \alpha\lambda^{-1} + \beta(\lambda^{-2} + \lambda),$$

where $r = \sqrt{x_1^2 + x_2^2}$. Hint: First integrate the relations of part (d) to determine p up to a function of time t, say $f(t)$, then apply the traction-free boundary condition to determine $f(t)$.

(f) Assuming further that the resultant forces on the end-faces of the cylinder are zero, derive the equation for λ:

$$\alpha\lambda^4 + \beta\lambda^3 - (\alpha - \frac{1}{4}\rho\omega^2 A^2)\lambda - \beta = 0.$$

Without solving this equation, what can you say about the roots? Noting that the length of the spinning cylinder is λL, what is happening to the cylinder physically as a result of the spinning?

<u>Problem 7.</u>

An elastic fluid is a continuum medium whose stored energy $e(\rho)$ (per unit mass) is a function of the density. Thus, the stored energy in a material volume $\mathcal{P}_0 \subset \Omega_0$ during a motion $\boldsymbol{x} = \boldsymbol{\varphi}_t(\boldsymbol{X})$ is:

$$\mathcal{E}(\mathcal{P}_t) = \int_{\mathcal{P}_t} e(\rho)\rho dv,$$

where $\mathcal{P}_t = \boldsymbol{\varphi}_t(\mathcal{P}_0)$. (Recall the constitutive relations for an elastic fluid $\boldsymbol{\sigma} = -p\mathbf{1}$.)

(a) Use the transport theorem and the balance of mass equation to compute the rate of change of $\mathcal{E}(\mathcal{P}_t)$.

(b) Let $P_{\text{int}}(\mathcal{P}_t)$ be the stress power. Use the result of part (a) to conclude that the constitutive relation for the pressure is

$$\frac{p}{\rho^2} = \frac{\mathrm{d}}{\mathrm{d}\rho}e(\rho).$$

(c) Define the enthalpy per unit mass as

$$h := e(\rho) + \frac{p}{\rho}.$$

Show that h is only a function of the pressure. Conclude that

$$\nabla h(p) = \frac{1}{\rho}\nabla p.$$

(d) Show that if the fluid is (a) steady and (b) irrotational, then

$$h(p) + \frac{1}{2}\|\boldsymbol{v}\|^2 + V = \text{constant},$$

where V is the potential of the external load.

(e) Apply the equation of part (d) to a fluid which flows from a large closed tank through a smooth thin pipe. If the pressure in the tank is N times the atmospheric pressure, determine the speed of the emerging fluid. Assume that the fluid satisfies the equation $p = \lambda\rho^k$, where λ and k are constants. Neglect gravity.

<u>Problem 8</u>

Recall the energy equation, or the First Law of Thermodynamics in the local form

$$\dot{e} = \boldsymbol{P}\colon\dot{\boldsymbol{F}} + r - \nabla\cdot\boldsymbol{q},$$

where e is the internal energy density, $\boldsymbol{P}$ is the first Piola-Kirchhoff stress, $\boldsymbol{F}$ is the deformation gradient, r is the heat source (heating per unit time per unit volume), and $\boldsymbol{q}$ is the heat flux vector (heat crossing a unit area per unit time). Its evil twin, the Second Law, can be written locally as

$$\dot{\eta} \geq \frac{r}{\theta} - \nabla\cdot\left(\frac{\boldsymbol{q}}{\theta}\right),$$

where η is the entropy density and θ is the temperature. These can be combined into the dissipation inequality, which upon assuming that $e = \hat{e}(\boldsymbol{F}, \eta)$ yields the general constitutive relations $\boldsymbol{P} = \partial e/\partial \boldsymbol{F}$ and $\theta = \partial e/\partial \eta$. The dissipation inequality also suggests the Fourier law of heat conduction: $\boldsymbol{q} = -\boldsymbol{K}\nabla\theta$, where $\boldsymbol{K}$ is the conductivity tensor. The Legendre transformation $\psi := e - \theta\eta$ defines the Helmholtz free energy density, where $\psi = \hat{\psi}(\boldsymbol{F}, \theta)$.

(a) Consider the following decoupled form of the Helmholtz free energy density: $\hat{\psi}(\boldsymbol{F}, \theta) = \hat{W}(\boldsymbol{F}) + \hat{\chi}(\theta)$. Use this to derive the heat equation from the energy equation laid out above. This form of the heat equation should explicity show the temperature rate, $\dot{\theta}$ driven by various mechanisms of heat supply. The coefficient of $\dot{\theta}$ in this form of the heat equation is the specific heat. Clearly indicate its general form.

(b) Now suppose that you have *coupled thermoelasticity*: $\hat{\psi}(\boldsymbol{F}, \theta) = \hat{W}(\boldsymbol{F} - \alpha(\theta - \theta_0)\mathbf{1}) + \hat{\chi}(\theta)$, where α is the coefficient of thermal expansion and θ_0 is a reference temperature. Now rederive the heat equation for $\dot{\theta}$. How is the heat equation in this case different from (a)?