Continuum Physics

Problem set 1

Problem 1.
Show that the vector \( ae_1 + be_2 + ce_3 \) is normal to the plane whose equation is \( ax + by + cz = d \).

Problem 2.
Show that \((a \times b) \times c = (a \cdot c)b - (b \cdot c)a\).

Problem 3.
Show that the permutation symbol, \( \varepsilon_{ijk} \) can be expressed as

\[
\varepsilon_{ijk} = \begin{vmatrix}
\delta_{i1} & \delta_{i2} & \delta_{i3} \\
\delta_{j1} & \delta_{j2} & \delta_{j3} \\
\delta_{k1} & \delta_{k2} & \delta_{k3}
\end{vmatrix}.
\]

Problem 4.
Prove the \( \varepsilon - \delta \) identity: \( \varepsilon_{ijk} \varepsilon_{irs} = \delta_{jr} \delta_{ks} - \delta_{js} \delta_{kr} \).

Problem 5.
If \( v \) is any vector and \( \hat{n} \) is any unit vector, show that \( v \) can be resolved into a component parallel to \( \hat{n} \) and a component perpendicular to it:

\[
v = (v \cdot \hat{n})\hat{n} + \hat{n} \times (v \times \hat{n}).
\]

Problem 6.
Prove the following, given that \( T \) is an arbitrarily chosen tensor of order 2:

(i) If \( S \) is a symmetric, second-order tensor, then \( S : T = S : T^T = S : \frac{1}{2}(T + T^T) \).

(ii) If \( W \) is a skew-symmetric, second-order tensor, then \( W : T = -W : T^T = W : \frac{1}{2}(T - T^T) \).

(iii) If \( S \) is symmetric and \( W \) is skew-symmetric, then \( S : W = 0 \).

Problem 7.
Let \( Q \) be an orthogonal tensor and let \( e \) be a vector such that \( Qe = e \).

(i) Show that \( Q^T e = e \).

(ii) Suppose \( W \) is a skew-symmetric tensor. Define its axial vector, \( \hat{w} \) by

\[
Wa = \hat{w} \times a
\]

for any vector \( a \). Determine the components of \( \hat{w} \) in terms of the components of \( W \).

(iii) Let \( \hat{w} \) be the axial vector corresponding to the skew part of \( Q \). Show that \( \hat{w} \) is parallel to \( e \).
Problem 8.
Show that if \( \hat{w} \) is the axial vector of a skew-symmetric tensor \( W \), then
\[
\| \hat{w} \| = \frac{1}{\sqrt{2}} \| W \|.
\]

Note: The Euclidean norm of a vector \( a \) is denoted \( \| a \| \), and is given by \( \| a \| = (a \cdot a)^{\frac{1}{2}} \). This is the same as the magnitude of \( a \), previously written as \( |a| \). Similarly, the Euclidean norm of a tensor \( A \) is given by \( \| A \| = (A : A)^{\frac{1}{2}} \).