Continuum Physics

Problem set 1

Problem 1.
Show that the vector $a\boldsymbol{e}_1 + b\boldsymbol{e}_2 + c\boldsymbol{e}_3$ is normal to the plane whose equation is $ax + by + cz = d$.

Problem 2.
Show that $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$.

Problem 3.
Show that the permutation symbol, ϵ_{ijk} can be expressed as

$$\epsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}.$$

Problem 4.
Prove the $\epsilon - \delta$ identity: $\epsilon_{ijk}\epsilon_{irs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}$.

Problem 5.
If $\boldsymbol{v}$ is any vector and $\hat{\boldsymbol{n}}$ is any unit vector, show that $\boldsymbol{v}$ can be resolved into a component parallel to $\hat{\boldsymbol{n}}$ and a component perpendicular to it:

$$\boldsymbol{v} = (\boldsymbol{v} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} + \hat{\boldsymbol{n}} \times (\boldsymbol{v} \times \hat{\boldsymbol{n}}).$$

Problem 6.
Prove the following, given that $\boldsymbol{T}$ is an arbitrarily chosen tensor of order 2:

(i) If $\boldsymbol{S}$ is a symmetric, second-order tensor, then $\boldsymbol{S}\colon\boldsymbol{T} = \boldsymbol{S}\colon\boldsymbol{T}^{\mathrm{T}} = \boldsymbol{S}\colon[\frac{1}{2}(\boldsymbol{T} + \boldsymbol{T}^{\mathrm{T}})]$.

(ii) If $\boldsymbol{W}$ is a skew-symmetric, second-order tensor, then $\boldsymbol{W}\colon\boldsymbol{T} = -\boldsymbol{W}\colon\boldsymbol{T}^{\mathrm{T}} = \boldsymbol{W}\colon[\frac{1}{2}(\boldsymbol{T} - \boldsymbol{T}^{\mathrm{T}})]$.

(iii) If $\boldsymbol{S}$ is symmetric and $\boldsymbol{W}$ is skew-symmetric, then $\boldsymbol{S}\colon\boldsymbol{W} = 0$.

Problem 7.
Let $\boldsymbol{Q}$ be an orthogonal tensor and let $\boldsymbol{e}$ be a vector such that $\boldsymbol{Q}\boldsymbol{e} = \boldsymbol{e}$.

(i) Show that $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{e} = \boldsymbol{e}$.

(ii) Suppose $\boldsymbol{W}$ is a skew-symmetric tensor. Define its *axial vector*, $\hat{\boldsymbol{w}}$ by

$$\boldsymbol{W}\boldsymbol{a} = \hat{\boldsymbol{w}} \times \boldsymbol{a}$$

for any vector $\boldsymbol{a}$. Determine the components of $\hat{\boldsymbol{w}}$ in terms of the components of $\boldsymbol{W}$.

(iii) Let $\hat{\boldsymbol{w}}$ be the axial vector corresponding to the skew part of $\boldsymbol{Q}$. Show that $\hat{\boldsymbol{w}}$ is parallel to $\boldsymbol{e}$.

Problem 8.

Show that if $\hat{\boldsymbol{w}}$ is the axial vector of a skew-symmetric tensor $\boldsymbol{W}$, then

$$\|\hat{\boldsymbol{w}}\| = \frac{1}{\sqrt{2}}\|\boldsymbol{W}\|.$$

Note: The *Euclidean norm* of a vector $\boldsymbol{a}$ is denoted $\|\boldsymbol{a}\|$, and is given by $\|\boldsymbol{a}\| = (\boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}}$. This is the same as the *magnitude* of $\boldsymbol{a}$, previously written as $|\boldsymbol{a}|$. Similarly, the *Euclidean norm* of a tensor $\boldsymbol{A}$ is given by $\|\boldsymbol{A}\| = (\boldsymbol{A}\colon\boldsymbol{A})^{\frac{1}{2}}$.