Continuum Physics

Problem set 2

Problem 1.

Consider a scalar field, ϕ, vector fields $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$, and a tensor field $\boldsymbol{A}$. Use indicial notation to show the following:

(i) $(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} \otimes \boldsymbol{d}$

(ii) $\nabla \times (\nabla \phi) = \boldsymbol{0}$

(iii) $\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}$, where $\nabla^2 \boldsymbol{v} := \nabla \cdot \nabla \boldsymbol{v}$

(iv) $\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$

(v) $\nabla(\phi \boldsymbol{v}) = \phi \nabla \boldsymbol{v} + \boldsymbol{v} \otimes \nabla \phi$

(vi) $\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = (\nabla \boldsymbol{v})^T \boldsymbol{u} + (\nabla \boldsymbol{u})^T \boldsymbol{v}$

(vii) $\text{div}(\phi \boldsymbol{A}) = \phi \text{div}(\boldsymbol{A}) + \boldsymbol{A} \nabla \phi$

(viii) $\text{div}(\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{u} \text{div}(\boldsymbol{v}) + (\nabla \boldsymbol{u}) \boldsymbol{v}$

Problem 2.

Let I_1, I_2, I_3 be the principal invariants of a tensor $\boldsymbol{A}$:

$$\begin{aligned} I_1 &= \text{trace}(\boldsymbol{A}) \\ I_2 &= \frac{1}{2}\left[(\text{trace}(\boldsymbol{A}))^2 - \text{trace}(\boldsymbol{A}^2)\right] \\ I_3 &= \det(\boldsymbol{A}). \end{aligned}$$

Show that these quantities are indeed invariant under an orthogonal transformation of the basis vectors, given by $\tilde{\boldsymbol{e}}_i = \boldsymbol{Q}\boldsymbol{e}_i$.

Problem 3.

New right-handed coordinate axes are chosen with the new origin at $(4, -1, -2)$ and with $\bar{\boldsymbol{e}}_1 = (2\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + \boldsymbol{e}_3)/3$ and $\bar{\boldsymbol{e}}_2 = (\boldsymbol{e}_1 - \boldsymbol{e}_2)/\sqrt{2}$.

(i) Express $\bar{\boldsymbol{e}}_3$ in terms of the $\boldsymbol{e}_k$.

(ii) If $\boldsymbol{t} = 10\boldsymbol{e}_1 + 10\boldsymbol{e}_2 - 20\boldsymbol{e}_3$, express $\boldsymbol{t}$ in terms of the new basis.

(iii) Express the old coordinates X_1, X_2, X_3 in terms of $\bar{X}_1, \bar{X}_2, \bar{X}_3$.

Problem 4.

A force of magnitude F acts in a direction radially away from the origin at a point $(a/3, 2b/3, 2c/3)$ on the surface of the ellipsoid $(X^2/a^2) + (Y^2/b^2) + (Z^2/c^2) = 1$. Determine the components of the normal to the surface, and hence the component of the force perpendicular to the surface at $(a/3, 2b/3, 2c/3)$.

Hint: Recall that the normal to a level surface $\phi(X, Y, Z) =$ constant is given by

$$\hat{\boldsymbol{n}} = \frac{\nabla\phi}{\|\nabla\phi\|}.$$

Problem 5.

A tensor $\boldsymbol{P}$ is a projection if $\boldsymbol{P}$ is symmetric and $\boldsymbol{P}^2 = \boldsymbol{P}$. Show that the following tensors are projections:

(i) $\mathbf{1}$

(ii) $\mathbf{0}$

(iii) $\boldsymbol{e} \otimes \boldsymbol{e}$, where $\boldsymbol{e}$ is any unit vector.

(iv) $\mathbf{1} - \boldsymbol{e} \otimes \boldsymbol{e}$

Problem 6.

Let $\boldsymbol{Q}$ be an orthogonal tensor, and $\boldsymbol{e}_1$ be of unit magnitude and such that $\boldsymbol{Q}\boldsymbol{e}_1 = \boldsymbol{e}_1$ and $\|\boldsymbol{e}_1\| = 1$. Also let $\boldsymbol{n}$ be any vector orthogonal to $\boldsymbol{e}_1$ and of unit magnitude: $\boldsymbol{n} \cdot \boldsymbol{e}_1 = 0, \|\boldsymbol{n}\| = 1$. Let $\boldsymbol{m}$ be such that $\boldsymbol{m} \cdot \boldsymbol{e}_1 = 0, \|\boldsymbol{m}\| = 1, \boldsymbol{m} = \boldsymbol{Q}\boldsymbol{n}$ and $\boldsymbol{m} \cdot \boldsymbol{n} = \cos\theta$.

(i) Sketch (graphically) the action of $\boldsymbol{Q}$ on $\boldsymbol{e}_1$ and $\boldsymbol{n}$.

(ii) Show that $\boldsymbol{Q}$ can be written as:

$$\boldsymbol{Q} = \mathbf{1}\cos\theta + (1 - \cos\theta)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - (\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 - \boldsymbol{e}_3 \otimes \boldsymbol{e}_2)\sin\theta,$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ form an orthonormal triad.

(iii) Show that

$$(\boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a})\boldsymbol{h} = -(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{h}.$$

(iv) Show that $\boldsymbol{Q}$ can also be written as:

$$\boldsymbol{Q} = \mathbf{1}\cos\theta + (1 - \cos\theta)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sin\theta\boldsymbol{E},$$

where $\boldsymbol{E}$ is the skew-symmetric tensor of which $\boldsymbol{e}_1$ is the axial vector.