Continuum Physics

Problem set 2

Problem 1.
Consider a scalar field, $\phi$, vector fields $u, v, a, b, c$ and $d$, and a tensor field $A$. Use indicial notation to show the following:

(i) $(a \otimes b)(c \otimes d) = (b \cdot c)a \otimes d$
(ii) $\nabla \times (\nabla \phi) = 0$
(iii) $\nabla \times (\nabla \times v) = \nabla(\nabla \cdot v) - \nabla^2 v$, where $\nabla^2 v := \nabla \cdot \nabla v$
(iv) $\nabla \cdot (\nabla \times v) = 0$
(v) $\nabla(\phi v) = \phi \nabla v + v \otimes \nabla \phi$
(vi) $\nabla(u \cdot v) = (\nabla v)^T u + (\nabla u)^T v$
(vii) $\text{div}(\phi A) = \phi \text{div}(A) + A \nabla \phi$
(viii) $\text{div}(u \otimes v) = u \text{div}(v) + (\nabla u)v$

Problem 2.
Let $I_1, I_2, I_3$ be the principal invariants of a tensor $A$:

\begin{align*}
I_1 &= \text{trace}(A) \\
I_2 &= \frac{1}{2} \left[ (\text{trace}(A))^2 - \text{trace}(A^2) \right] \\
I_3 &= \text{det}(A).
\end{align*}

Show that these quantities are indeed invariant under an orthogonal transformation of the basis vectors, given by $\tilde{e}_i = Qe_i$.

Problem 3.
New right-handed coordinate axes are chosen with the new origin at $(4, -1, -2)$ and with $\tilde{e}_1 = (2e_1 + 2e_2 + e_3)/3$ and $\tilde{e}_2 = (e_1 - e_2)/\sqrt{2}$.

(i) Express $\tilde{e}_3$ in terms of the $e_k$.
(ii) If $t = 10e_1 + 10e_2 - 20e_3$, express $t$ in terms of the new basis.
(iii) Express the old coordinates $X_1, X_2, X_3$ in terms of $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$.

Problem 4.
A force of magnitude $F$ acts in a direction radially away from the origin at a point $(a/3, 2b/3, 2c/3)$ on the surface of the ellipsoid $(X^2/a^2) + (Y^2/b^2) + (Z^2/c^2) = 1$. Determine the components of the normal to the surface, and hence the component of the force perpendicular to the surface at $(a/3, 2b/3, 2c/3)$.
Hint: Recall that the normal to a level surface $\phi(X, Y, Z) = \text{constant}$ is given by
\[
\hat{n} = \frac{\nabla \phi}{\|\nabla \phi\|}.
\]

Problem 5.
A tensor $P$ is a projection if $P$ is symmetric and $P^2 = P$. Show that the following tensors are projections:

(i) $1$

(ii) $0$

(iii) $e \otimes e$, where $e$ is any unit vector.

(iv) $1 - e \otimes e$

Problem 6.
Let $Q$ be an orthogonal tensor, and $e_1$ be of unit magnitude and such that $Qe_1 = e_1$ and $\|e_1\| = 1$. Also let $n$ be any vector orthogonal to $e_1$ and of unit magnitude: $n \cdot e_1 = 0, \|n\| = 1$. Let $m$ be such that $m \cdot e_1 = 0, \|m\| = 1, m = Qn$ and $m \cdot n = \cos \theta$.

(i) Sketch (graphically) the action of $Q$ on $e_1$ and $n$.

(ii) Show that $Q$ can be written as:
\[
Q = 1 \cos \theta + (1 - \cos \theta)e_1 \otimes e_1 - (e_2 \otimes e_3 - e_3 \otimes e_2)\sin \theta,
\]
where $\{e_1, e_2, e_3\}$ form an orthonormal triad.

(iii) Show that
\[
(a \otimes b - b \otimes a)h = -(a \times b) \times h.
\]

(iv) Show that $Q$ can also be written as:
\[
Q = 1 \cos \theta + (1 - \cos \theta)e_1 \otimes e_1 + \sin \theta E,
\]
where $E$ is the skew-symmetric tensor of which $e_1$ is the axial vector.