Continuum Physics

Problem set 3

Problem 1.

In a region of space the flow velocity components are:

$$v_1 = A(x_1^3 + x_1 x_2^2)e^{-kt}, \quad v_2 = A(x_1^2 x_2 + x_2^3)e^{-kt}, \quad v_3 = 0,$$

where A and k are given constants, x_1, x_2, x_3 are spatial coordinates and t is time. Find the acceleration components at the point $(1, 1, 0)$ at time $t = 0$.

Problem 2.

The motion of a continuous medium is defined by the equations

$$\begin{aligned} x_1 &= \frac{1}{2}(X_1 + X_2)e^t + \frac{1}{2}(X_1 - X_2)e^{-t} \\ x_2 &= \frac{1}{2}(X_1 + X_2)e^t - \frac{1}{2}(X_1 - X_2)e^{-t} \\ x_3 &= X_3 \end{aligned}$$

(a) Express the velocity components in terms of the material coordinates and time.

(b) Express the velocity components in terms of the spatial coordinates and time.

(c) Calculate the components of the rate-of-deformation tensor $\boldsymbol{d}$.

(d) Express the displacement components, u_1, u_2, u_3 in terms of the material coordinates and time, where $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$.

(e) The *infinitesimal strain* tensor is

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{F} + \boldsymbol{F}^{\mathrm{T}}) - \mathbf{1}.$$

Express the components of $\boldsymbol{\varepsilon}$ in terms of material coordinates and time. Evaluate them at $t = 0$ and $t = 0.05$.

(f) Calculate the rate of change $\mathrm{d}\varepsilon_{ij}/\mathrm{d}t$ of the infinitesimal strain components and compare them with d_{ij} at $t = 0$ and $t = 0.05$. Comment upon the differences.

Problem 3.

Consider a rigid body motion in which $\boldsymbol{F} = \boldsymbol{R}(t)$, where $\boldsymbol{R} \in \mathrm{SO}(3)$. Evaluate the infinitesimal strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{F} + \boldsymbol{F}^{\mathrm{T}} - 2\mathbf{1})$ and the Lagrange strain tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \mathbf{1})$ in this case. What are the differences? Comment upon the difference between $\boldsymbol{\varepsilon}$ and $\boldsymbol{E}$ for

$$\boldsymbol{R}(t) = \mathbf{1}\cos t + (1 - \cos t)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - (\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 - \boldsymbol{e}_3 \otimes \boldsymbol{e}_2)\sin t, \quad \text{for } t = 0, 0.01\pi, 0.1\pi, 0.2\pi, 0.3\pi, 0.4\pi, 0.5\pi$$

Problem 4.

In a region of space we have steady fluid flow given by

$$v_1 = -wx_2 - \frac{Ax_2}{x_1^2 + x_2^2}, \quad v_2 = wx_1 + \frac{Ax_1}{x_1^2 + x_2^2}, \quad v_3 = 0,$$

where w and A are constants. Write out the matrices of the rate of deformation tensor, $\boldsymbol{d}$ and the spin tensor, $\boldsymbol{w}$ at a point (x_1, x_2, x_3).

Problem 5.

Simple shear deformation can be represented as

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3,$$

where γ is a positive constant.

(a) Sketch the initial and final configurations of a element that was originally a square $ABCD$ in the X_1X_2-plane, of side length dL, with AB parallel to the X_1-axis and AD to the X_2-axis.

(b) Show that the deformation gradient can be written as $\boldsymbol{F} = \boldsymbol{1} + \gamma \boldsymbol{l} \otimes \boldsymbol{m}$, where $\boldsymbol{l}$ and $\boldsymbol{m}$ are orthonormal. Write out the components in matrix form.

(c) Show that the right Cauchy-Green tensor $\boldsymbol{C}$ can be written as

$$\boldsymbol{C} = \boldsymbol{l} \otimes \boldsymbol{l} + (1 + \gamma^2)\boldsymbol{m} \otimes \boldsymbol{m} + \boldsymbol{n} \otimes \boldsymbol{n} + \gamma(\boldsymbol{l} \otimes \boldsymbol{m} + \boldsymbol{m} \otimes \boldsymbol{l}),$$

where $\{\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}\}$ form an orthonormal triad. Write the components of $\boldsymbol{C}$ in matrix form.

(d) Show that the leftt Cauchy-Green tensor $\boldsymbol{b}$ can be written as

$$\boldsymbol{b} = (1 + \gamma^2)\boldsymbol{l} \otimes \boldsymbol{l} + \boldsymbol{m} \otimes \boldsymbol{m} + \boldsymbol{n} \otimes \boldsymbol{n} + \gamma(\boldsymbol{l}\boldsymbol{m} + \boldsymbol{m} \otimes \boldsymbol{l}).$$

Write the components of $\boldsymbol{b}$ in matrix form.

(e) Determine the principal stretches $\lambda_A, (A = 1, 2, 3)$.

(f) Determine the referential stretch axes (i.e., the normalized eigenvectors of $\boldsymbol{C}$), also called the Lagrangian triad.

(g) Determine the current stretch axes (i.e., the normalized eigenvectors of $\boldsymbol{b}$), also called the Eulerian triad..

(h) Determine the rotation tensor, $\boldsymbol{R}$.

(i) If $\boldsymbol{M}_A$ is an eigenvector of $\boldsymbol{C}$, and $\boldsymbol{m}_A$ is the corresponding eigenvector of $\boldsymbol{b}$, show that $\boldsymbol{m}_A = \boldsymbol{R}\boldsymbol{M}_A$, for $A = 1, 2, 3$.

Problem 6.

A motion in which the spatial velocity is given by

$$\boldsymbol{v} = c(\boldsymbol{p} \otimes \boldsymbol{q})\boldsymbol{x} = c(\boldsymbol{x} \cdot \boldsymbol{q})\boldsymbol{p},$$

where c is a positive constant and $\boldsymbol{p}, \boldsymbol{q}$ are orthogonal unit vectors, is called a *simple shearing motion.* For this motion:

(a) Express the rate of deformation tensor, $\boldsymbol{d}$ and the spin tensor, $\boldsymbol{w}$ in terms of $\boldsymbol{p}, \boldsymbol{q}$ and c.

(b) Write the spectral decomposition of $\boldsymbol{d}$.

(c) Determine the principal stretchings of the motion: i.e., the eigenvalues of $\boldsymbol{d}$.

(d) Determine the principal axes of stretching: i.e., the eigenvectors of $\boldsymbol{d}$.

(e) Determine the angular velocity of the motion: i.e., the axial vector of the spin, $\boldsymbol{w}$.