Continuum Physics

Problem set 4

Problem 1.

(a) Let $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ be skew-symmetric tensors with axial vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$. Show that

$$\boldsymbol{W}_1\boldsymbol{W}_2 = \boldsymbol{w}_2 \otimes \boldsymbol{w}_1 - (\boldsymbol{w}_1 \cdot \boldsymbol{w}_2)\mathbf{1}$$

(b) Let $\boldsymbol{W}$ be an arbitrary skew-symmetric tensor with axial vector $\boldsymbol{w} \in \mathbb{R}^3$. Find the expression for $\boldsymbol{W}^3$ and, in general for $\boldsymbol{W}^N$ where $N > 0$ is any positive integer, in terms of $\boldsymbol{W}$, $\boldsymbol{W}^2$ and the powers of $\|\boldsymbol{w}\|$.

Problem 2.

Obtain the conditions on the deformation gradient ensuring that, at a given particle:

(a) No extension occurs in a specified direction $\boldsymbol{M} \in \mathbb{R}^3$.

(b) The angle between a specified pair of directions, say $\boldsymbol{M}_1, \boldsymbol{M}_2 \in \mathbb{R}^3$ remains unchanged ($\boldsymbol{M}_1, \boldsymbol{M}_2$ need not be orthogonal).

(c) No change in area takes place in the plane perpendicular to a given direction.

(d) No change in volume occurs.

Problem 3.

The (spatial) velocity of a body, $\boldsymbol{v}(\boldsymbol{x}, t)$ is:

$$v_1 = \frac{-n\sin(nt)}{2+\cos(nt)}x_1, \quad v_2 = \frac{n\cos(nt)}{2+\sin(nt)}x_2, \quad v_3 = 0,$$

where n is an integer.

(a) Compute the rate of deformation tensor, $\boldsymbol{d}(\boldsymbol{x}, t)$, the spin tensor, $\boldsymbol{w}(\boldsymbol{x}, t)$, and the vorticity vector, $\hat{\boldsymbol{\omega}}(\boldsymbol{x}, t)$.

(b) Choosing as reference configuration the placement of the body at time $t = 0$, obtain the particle paths (i.e., the motion $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t)$.

(c) The material velocity field $\boldsymbol{V}(\boldsymbol{X}, t)$.

Problem 4.

Consider a motion $\boldsymbol{\varphi}(\boldsymbol{X}, t)$ with deformation gradient $\boldsymbol{F}$ relative to coincident spatial and material reference frames:

$$\boldsymbol{F}(\boldsymbol{X}, t) = \alpha_1(t)\boldsymbol{a}_1(t) \otimes \boldsymbol{e}_1 + \alpha_2(t)\boldsymbol{a}_2(t) \otimes \boldsymbol{e}_2 + \alpha_3(t)\boldsymbol{a}_3(t) \otimes \boldsymbol{e}_3$$

where the vectors $\{\boldsymbol{a}_i(t), i = 1, 2, 3\}$ are defined as

$$\boldsymbol{a}_i = \boldsymbol{Q}(t)\boldsymbol{e}_i, \quad [\boldsymbol{Q}(t)]^{\mathrm{T}}\boldsymbol{Q}(t) = \mathbf{1}, \quad (i = 1, 2, 3)$$

(a) Find the right and left Cauchy-Green tensors, and the Lagrangian strain tensor, $\boldsymbol{E} = \frac{1}{2}[\boldsymbol{C} - \boldsymbol{1}]$.

(b) Find the polar decomposition of $\boldsymbol{F}$.

(c) Find the rate of deformation tensor, $\boldsymbol{d}$, the spin tensor, $\boldsymbol{w}$ and the vorticity vector, $\hat{\boldsymbol{\omega}}$.

(d) Let $\Omega(\boldsymbol{X}, t)$ be defined in general as:

$$\Omega := \frac{\partial \boldsymbol{R}}{\partial t}\boldsymbol{R}^{\mathrm{T}},$$

where $\boldsymbol{R}$ is the rotation tensor. Find the general expression relating Ω and the spin tensor, $\boldsymbol{w}$. Apply your result to the particular case given by $\boldsymbol{F}$ above.

Problem 5.

Consider the radial motion of a cylinder with axis along $\boldsymbol{e}_3$ (basis vector along X_3) and the inner radius $A > 0$, defined by

$$\begin{aligned} x_1 &= \varphi_1(X_1, X_2, X_3, t) := \frac{g(R,t)}{R}X_1, \\ x_2 &= \varphi_2(X_1, X_2, X_3, t) := \frac{g(R,t)}{R}X_2, \\ x_3 &= \varphi_3(X_1, X_2, X_3, t) := \frac{g(R,t)}{R}X_3, \end{aligned}$$

where $R := \sqrt{X_1^2 + X_2^2}$, and the reference position $\boldsymbol{X} \in \Omega_0$ and the current position $\boldsymbol{x} \in \Omega_t$ are taken relative to a cartesian reference frame.

(a) Compute $\boldsymbol{F}$

(b) Compute J

(c) Assuming that the motion is isochoric (i.e., volume-preserving), find $g(R, t)$.