Continuum Physics

Problem set 5

The first two problems in this set extend fundamental ideas a bit further. The next three are supposed to help you visualize basic concepts in stress analysis: stress distributions in particular. The last three are on ideas in fluid mechanics.

Problem 1.

To fix ideas suppose that the continuum body of interest is a fluid. A control volume is a region $\mathcal{P}$, *fixed in space*. Therefore as time changes fluid particles enter and leave the region $\mathcal{P}$. Consider a motion $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t)$ and suppose that the density and velocity fields in the spatial description are $\rho(\boldsymbol{x}, t)$ and $\boldsymbol{v}(\boldsymbol{x}, t)$ respectively.

(a) Establish the integral form of the balance of mass for the control volume $\mathcal{P}$; in other words, what does $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \rho(\boldsymbol{x}, t)dv$ equal in terms of the spatial field? Express the result in terms of the surface integral over the boundary, with outward normal $\boldsymbol{n}$.

(b) Establish the integral form of the conservation law for linear momentum for the control volume $\mathcal{P}$; in other words, what is $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)dv$?

Problem 2.

Consider a motion $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t)$, with spatial density $\rho(\boldsymbol{x}, t)$, spatial velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ and Cauchy stress tensor $\boldsymbol{\sigma}(\boldsymbol{x}, t)$. Show that the balance of mass and linear momentum can be written as the first order system

$$\frac{\partial \rho}{\partial t} = \mathrm{div}[\rho \boldsymbol{v}]$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} = \mathrm{div}[\boldsymbol{\sigma} - \rho \boldsymbol{v} \otimes \boldsymbol{v}] + \boldsymbol{b}.$$

How are these results related to those in Problem 1?

Remark: These equations are also called the conservation form of the balance laws. They play a central role in the design of numerical methods for fluids. For classical finite difference methods, this conservation form is preferred over the form presented in class [for continuum problems, of course they are identical]. Finite element methods, however, can be formulated with equal ease for either form.

Problem 3.

Calculate and plot the surface tractions obtained from the Cauchy stress tensor on the surface of a sphere, cube and a tetrahedron if

(a) The Cauchy stress is $\boldsymbol{\sigma} = -p\mathbf{1}$, with $p =$ const.. Interpret the result.

(b) The Cauchy stress has components relative to a Cartesian coordinate system $\sigma_{ij} = a$, for $i, j = 1, 2, 3$, with $a =$ const..

Problem 4.

Consider a continuum body with reference configuration Ω_0, whose current configuration at time t is a parallelopiped defined relative to a Cartesian reference system according to

$$\Omega_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_1 \leq l,\ -c \leq x_2 \leq c,\ -h \leq x_3 \leq h\}.$$

The Cauchy stress has been measured experimentally and is found to be closely replicated by the following expressions:

$$\sigma_{11} = (3P/2c^3)(l - x_1)x_2, \quad \sigma_{12} = -(3P/4c)(1 - x_2^2/c^2),$$

where P is a constant and $\sigma_{22} = \sigma_{13} = \sigma_{23} = \sigma_{33} = 0$.

(a) Determine the surface tractions on each of the six boundaries of Ω_t. Sketch your results. It is enough to sketch the results in the plane $x_1 - x_2$. Why?

(b) Under what assumption, if any, are the equations of balance of linear momentum satisfied?

Problem 5.

Suppose that the state of the Cauchy stress in the current configuration of a body relative to a Cartesian system $x_1 = x,\ x_2 = y,\ x_3 = z$, is *plane*: $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$. Suppose further that there are no body forces and that the spatial velocity field, $\boldsymbol{v}(\boldsymbol{x}, t)$ vanishes. Finally suppose that the non-zero components of the stress field are given by

$$\sigma_{xx} = \frac{\partial^2 F(x,y)}{\partial^2 y}, \ \sigma_{xy} = -\frac{\partial^2 F(x,y)}{\partial x \partial y}, \text{ and } \sigma_{yy} = \frac{\partial^2 F(x,y)}{\partial^2 x},$$

in terms of a given function $F(x, y)$. Such a function is called the *Airy's stress function.*

(a) Show that balance of linear momentum is identically satisfied by the above stress field.

(b) Consider a polar coordinate system, defined by the conventional relations

$$x = r\cos(\vartheta), \quad y = r\sin(\vartheta),$$

and suppose that the Airy stress function is

$$F(r, \vartheta) = C[r^2(\alpha - \vartheta) + r^2 \cos(\vartheta)\sin(\vartheta) - r^2 \cos^2(\vartheta)\tan(\alpha)],$$

where C and α are constants. Determine the stress distributions.

(c) Consider an infinite wedge at the origin O in the current configuration, limited by the plane Oxz with unit outward normal $\boldsymbol{n}_1 = \langle 0, -1, 0\rangle^T$, and a plane with unit outward normal $\boldsymbol{n}_2 = \langle -\sin(\alpha), \cos(\alpha), 0\rangle^T$ (both relative to the Cartesian system $\{x, y, z\}$). What are the tractions on these planes? Sketch them.

Remark: Airy stress functions were widely used in classical elasticity. The advent of the finite element method has virtually put them out of use in analysis.

Problem 6.

(a) Let $\boldsymbol{a} = \boldsymbol{M}\boldsymbol{A}\boldsymbol{M}^{\mathrm{T}}$, where $\boldsymbol{M}$ is an arbitrary tensor, $\boldsymbol{a}$ and $\boldsymbol{A}$ are skew-symmetric tensors with corresponding axial vectors $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{A}}$. Show that $\hat{\boldsymbol{a}} = \det(\boldsymbol{M})\boldsymbol{M}^{-\mathrm{T}}\hat{\boldsymbol{A}}$.

(b) Show that $\boldsymbol{AB} - \boldsymbol{BA} = \boldsymbol{D}$ where $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{D}$ are skew-symmetric tensors with corresponding axial vectors $\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{A}} \times \hat{\boldsymbol{B}}$.

Problem 7.

An ideal fluid is a continuum that does not support shear stress, i.e., $\boldsymbol{\sigma} = -p\mathbf{1}$.

(a) What does the statement of expended power become for this material?

(b) An incompressible fluid is one for which $\mathrm{div}(\boldsymbol{v}) = 0$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} K(\Omega_t) = P_{\mathrm{ext}}(\Omega_t).$$

Conclude that if there are no forces the kinetic energy is conserved.

Problem 8.

Show that the spatial accelration can be written as

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \mathrm{curl}(\boldsymbol{v}) \times \boldsymbol{v} + \frac{1}{2}\nabla \|\boldsymbol{v}\|^2.$$

Hint: Write the convective term as

$$(\nabla \boldsymbol{v})\boldsymbol{v} = [\nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^{\mathrm{T}}]\boldsymbol{v} + (\nabla \boldsymbol{v})^{\mathrm{T}}\boldsymbol{v}$$

and use the definition of vorticity.

Problem 9.

Consider an incompressible fluid with constant density.

(a) Show that the equations of motion are given by

$$\mathrm{curl}(\boldsymbol{v}) \times \boldsymbol{v} + \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla(V + \frac{p}{\rho} + \frac{1}{2}\|\boldsymbol{v}\|^2)$$

where $\frac{\boldsymbol{b}}{\rho} = -\nabla V$.

(b) An irrotational fluid is one for which $\mathrm{curl}(\boldsymbol{v}) = 0$. A steady flow is one for which $\partial \boldsymbol{v}/\partial t = \mathbf{0}$. Show that for a steady, irrotational, incomrpessible flow, subject to conservative body forces, we have

$$V + \frac{p}{\rho} + \frac{1}{2}\|\boldsymbol{v}\|^2 = \text{constant}.$$

This is the classical Bernoulli equation for this flow.